THE

MATHEMATICAL GAZETTE

EDITED BY

T. A. A. BROADBENT, M.A.

ROYAL NAVAL COLLEGE, GREENWICH, LONDON, S.E. 10

LONDON

G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXXV | DECEMBER, 1951 | No. 314

THE MATHEMATICS OF EASTER.

BY E. J. F. PRIMROSE.

EASTER SUNDAY is defined as the first Sunday after the Paschal full moon, which is the first full moon after 21st March. In practice, Easter Sunday is determined by means of a formula given in the Prayer Book. There also exists a formula due to Gauss. It is interesting to investigate the basis of these formulae and to compare their accuracy.

Gauss' formula

The Gauss formula for 1900–1999 is *

(i) Let the last two digits of the year-number be called x.

(ii) Let the remainder on dividing x by 4 be called a.

(iii) Let the remainder on dividing x by 7 be called b.

(iv) Take the remainder on dividing x by 19, multiply it by 19, add 24, divide by 30, and let the remainder be called c.

(v) Let the remainder on dividing $2a+4b+6c+3$ by 7 be called d.

Then Easter Sunday is $c+d$ days after 22nd March.

In this formula, c determines the full moon, and d determines the following Sunday. It should be noted that if $d=0$, the full moon falls on a Sunday (according to the formula) and the formula gives Easter Sunday as *that day*, not the following Sunday.

Theoretical basis

We first need to find a cycle of years, not too large, such that the Paschal full moons at the beginning and end of the cycle occur at about the same time on the same date. Now the time between two full moons is 29·53059 days (to 5 places) and a year is 365·2422 days (to 4 places), though of course we use

* The formula can clearly be extended to cover any period of 100 years.

years of 365 or 366 days, in practice. Expressing 29·53059/365·2422 as a continued fraction, we get

$$\frac{1}{12+}\;\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{17+}\;\cdots.$$

We therefore obtain a good approximation by stopping just before the 17, which gives us 19/235. In fact, 235 full moon periods exceed 19 years by only ·077 of a day.

Now, 13 full moon periods amount to 383·8977 days (to 4 places), so that, comparing two successive years, the full moon is 18·8977 days later in the second year, unless this is a leap year, in which case it is 17·8977 days later. (If this gives us the wrong full moon, we subtract 29·5306 days to give us the right one.) Now, in a practical formula, we can only work with integers, so we assume that the difference each year is 19 days, ignoring leap years completely, and subtract 30 days if this gives the wrong full moon. It will be seen that after a cycle of 19 years, we shall have to make an adjustment of 1 day, since $19\times 19-12\times 30=1$. In effect, this means that the last difference is only 18 days.

We can now see the reason for the formula for c. Counting days after 22nd March, in 1900 the Paschal full moon was 24. During the next 19 years we add 19 days each year and subtract 30 when necessary. After this we start the cycle of 19 years over again.

To find the following Sunday, we use the formula below.* To find the day of the week of any date from 1900 to 1999 we add together

(i) the last two digits of the year-number (x),
(ii) $\frac{1}{4}x$, neglecting fractions,
(iii) an appropriate figure for the month, which is 4 for March,
(iv) the date in the month.

We divide the sum by 7, and the remainder gives the day of the week (Sunday 1, ... , Friday 6, Saturday 0). In the subsequent work, we shall get dates in April, but clearly, for this purpose April $y=$ March $(31+y)$.

From the Gauss formula,

$$x=4m+a=7n+b,$$

where m and n are integers. So the day of the week of March $(22+c+d)$ is given by

$$\begin{aligned}&(x+m+4+22+c+2a+4b+6c+3) && \text{mod. } 7\\ &\equiv\{x+m+2(x-4m)+4x+1\} && \text{mod. } 7\\ &\equiv 1 && \text{mod. } 7.\end{aligned}$$

This verifies that the addition of d to the date of the full moon gives us the following Sunday (except for $d=0$, as mentioned above).

Prayer Book formula

This formula uses the same cycle of 19 years, and the same difference of 19 days each year, with two curious exceptions, that 5 and 16 years after the beginning of the cycle,† the addition is only 18 days, the next one being 20 days. In all normal years the formula gives the full moon one day earlier than the Gauss formula does, but in the exceptional years it is two days earlier. The Sunday is then calculated by essentially the same method as in Russell's formula, except that if the full moon occurs on a Sunday we add 7.

* This and the Gauss formula are given in *Rapid Calculations* by A. H. Russell.

† We take 1900 as the beginning of one cycle.

In the formulae for the full moon the difference of one day is understandable, because if the full moon occurs on a Sunday according to the Prayer Book, then Easter is taken to be the following Sunday : in this case, the Gauss formula takes the full moon as Monday and also makes Easter the following Sunday. In all other cases the difference obviously does not affect the Sunday required. However, the two days difference seems quite inexplicable, and, as we shall see, leads to errors. The two formulae agree on Easter Sunday for every year except 1954 and 1981.

The following table compares the actual Prayer Book and Gauss full moons for a cycle of 19 years and other interesting years. As before, the numbers refer to days after 22nd March, and 27·50, for example, means midday on 18th April. The day of the week refers to the actual full moon.

Year	*Actual*	*Day of Week*	*Prayer Book*	*Gauss*
1900	24·07	Su	23	24
1938	24·44	F.	23	24
39	13·81	Tu.	12	13
40	2·18	Su.	1	2
41	21·08	Sa.	20	21
42	10·44	W.	9	10
43	**29·34**	Tu.	**27**	**29**
44	17·71	Sa.	17	18
45	7·07	Th.	6	7
46	25·97	Tu.	25	26
47	15·34	Su.	14	15
48	3·71	Th.	3	4
49	22·60	W.	22	23
50	11·97	Su.	11	12
51	1·34	Sa.	0	1
52	19·24	Th.	19	20
53	8·60	M.	8	9
54	**27·50**	Su.	**26**	**28**
55	16·87	Th.	16	17
56	6·23	W.	5	6
1981	**0·18**	Su.	**27**	**29**
2190	**29·30**	F.	**27**	**29**

It will be observed that in 1943 (one of the exceptional years in the cycle) the Prayer Book formula gives the full moon two days early : this does not matter because it is actually on a Tuesday. In 1954 the full moon is actually on a Sunday, so Easter should be on the following Sunday, as given by the Gauss formula (which gives the full moon on a Monday), but the Prayer Book gives the full moon on a Saturday, so Easter is a week early. 1981 is a most interesting year, because both formulae give the *wrong full moon*. However, taking the following full moon, which occurs on a Monday, again the Gauss formula is right and the Prayer Book wrong.

The Prayer Book formula is altered every 300 years to allow for the error of about one day which has arisen by then. The period covered by the present formula is 1900–2199. The year 2190 was worked out to see whether the exceptional years in the cycle are given more accurately at the end of this period, but this appears not to be the case.

E. J. F. P.

TRIGONOMETRY IN THE MAIN SCHOOL.*

Mr. C. O. Tuckey : The reason why the programme committee invited me to open this discussion, instead of inviting someone still dusty with the chalk of a mathematical class-room, was doubtless because I had the honour to be chairman of the committee that drew up the *Trigonometry Report* published last August. So you will not be surprised if I make that *Report* the centre of my remarks. In fact, the advice I have to give on the teaching of trigonometry in the main school may be summed up by saying that :

" I advise all young teachers and most old ones to read the *Report* and act upon the suggestions it contains."

I suppose, however, that those who made me stand up here would regard this as too short an introduction to Mr. Parsons' remarks, so I shall pick out and call your attention to some of the features of that *Report*, and perhaps digress a little.

To begin with I will say that if anything in the *Report* meets with disapproval, it is not for want of it having been discussed in committee. Some of you would perhaps be surprised to find out how much argument can centre round the question " which is the best proof of the addition theorem " or " which should come first, the sine or the tangent ". Even the position of a pair of commas may keep a committee busy for some time. Of course, commas do really make a difference ; the simple statement " A says B is wrong " can be contradicted in precisely the same words in precisely the same order by the introduction of a pair of commas after A and B, so that " A, says B, is wrong ".

You realise that in these committees, when the subject to be treated has been divided into individual items, a set of scripts on these items are got ready, written by various members of the committee, and these scripts are later pulled to pieces by the committee as a whole. The order in which the items are discussed is not necessarily the order in which they will be printed. It depends largely on who gets his homework done first. For instance, in this case we began by discussing various proofs of the addition theorems, and only got back to the start later.

The history of the scripts on the addition theorems is typical and possibly instructive. Several members of the committee had their own favourite methods. The difficulty, of course, is that proofs for the case when the sum of two acute angles is also an acute angle do not, without considerable precautions, fit other cases. We have set out several limited proofs in the *Report*, but for the general case one man advocated very strongly the proof by projection as the best method and thought his proof covered all cases. After other members had produced diagrams which his proof did not fit, he thought again, and for a later meeting produced a proof beautifully complete, with charming diagrams in colours to show that all cases were covered : but, alas, his script occupied about six pages. Unfortunately, we had decided that eight methods were worth giving, and to give so much space to this one was to give it too much. So all the beautiful coloured diagrams were wasted, and we struggled to shorten the proof, so that finally it appears, fully general, in about a page and a half. Now you may feel that all this has not much bearing on actual teaching practice. But if you feel this way, you are wrong. If a method requires such a lot of discussion in committee in order to make it at once complete and reasonably short, it is safe to conclude that it is not as easy for the beginner as its advocates might suppose. That is why this method is labelled " a valuable method for the abler pupils ".

* Discussion at the Annual Meeting of the Mathematical Association, Bristol, 1951.

Another method, the one we recommend as " a very good general method ", illustrates rather strikingly the dependence of one thing on another in teaching. It is the method (Fig. 1) where PQ^2, found from the triangle OPQ, is

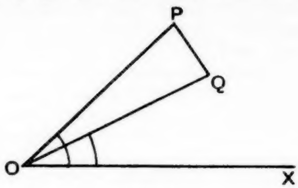


FIG. 1.

equated to its value found from the coordinates of P and Q. Though this seems an obvious method nowadays, it was not available forty years ago, for at that time pupils had to tackle the addition theorems *before* they were familiar with the distance formula of coordinate geometry. As the use of graphs at an early age increased, this order of teaching was reversed, and the method crept into the textbooks. As far as I know, it appeared first to prove the corresponding formula in solid geometry,

$$\cos \theta = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma',$$

in a book by two authors, who were stupid enough not to realise that they had got hold of a new and better proof for $\cos (A - B)$, and allowed someone else to get that proof into print first. Doubtless, however, the proof is more properly attributed to Ptolemy.

Two more points and I will leave the addition theorems. One is that the formulae for the general angle, $\sin (180° \pm \theta)$ in terms of $\sin \theta$, etc., needed for the addition theorems are set out in Chap. VII, p. 61 of the *Report*, in a comprehensive and convenient manner, which may easily be new to some teachers. The other point is this. We have recently been seeing and are still debating the fusion of trigonometry with geometry. Supposing that another twenty-five years may see the fusion of both these subjects with vector analysis, the best proof of the addition theorems will then be that given as the last of our proofs. It occupies one-third of a page.

To come back to the beginning of the subject. Young teachers who are, of course, familiar with the six trigonometrical ratios, will probably feel surprised at first to see how easily these ratios are muddled by their pupils. This often leads to demands for mnemonics. I wonder what type of mnemonic you prefer. Do you like little rhymes such as

" Remember the sine gives the height,
But the cosine the step to the right ",

or do you prefer a mysterious word, suggesting a volcano in Mexico or the Pacific, like

SOHCAHTOA.

Incidentally, at an examiners' meeting an assistant was heard saying to the chief : " What does this fellow mean by writing SOHCAHTOA at the top of every page? It looks most insulting." Or perhaps you are one of those who prefer to use perpendicular and base instead of opposite and adjacent, and who like to recall the great truth that

" Some people have, curly brown hair, till painted black ",

with its inevitable sequel that

" Such people have, curly black hair, till perfectly bald."

If you prefer this course, who shall blame you?

However, if the beginnings are properly taught, then there is little need for mnemonics. The proper way is to begin with one ratio only, and to go on with that one till it is quite firmly known, and only then to bring in the next. So far the committee were all agreed, but the question then arose, " What ratio should be recommended for a start? " The cotangent, secant and cosecant have no backers, and few will advise the cosine, except perhaps for rather older pupils. The main argument against putting the cosine before the sine is the same as that which prevents anyone from putting the cotangent before the tangent. There has, however, always been and probably will long continue to be, argument between those who favour the sine and those who favour the tangent.

Personally I favour the sine. Its definition is so easily extended to obtuse angles and it leads so quickly to the sine rule with its many consequences, but I shall be surprised and disappointed if no one stands up a little later on to explain why he or she favours the tangent. I will grant to him or her in advance that fifteen years ago an excellent case could have been made out for the tangent. But since then two things have happened to boost the sine, and incidentally to illustrate how one part of the teaching programme depends on another. The first of these is the fusion, if that is not too strong a word, of trigonometry with geometry. As Mr. Parsons is going to concentrate on this aspect of the matter, I will only say that in the usual figures of geometry the sine is more useful than the tangent. The second is that nowadays a considerable percentage of those who learn trigonometry do so, not to be able to solve triangles, but to be able to use the sine curve in the study of oscillations, in electricity, radio, etc. This fact should, I feel, affect teachers in grammar schools as well as those in technical colleges. The *Report*, in Chap. IV, discusses the necessary treatment of the sine curve.

In Chap. VI, on Field Work, an account is given of how a plane table can be made, and a set of suitable problems for its use follow. Of these, the method of resection from three points is that least well known, and is of value at various stages. In this chapter you can also learn how to make a sun-dial.

These discussions are of most value to those teachers who not only have convenient space for outdoor work, but are clever with their fingers at producing home-made apparatus. Since I was never of this number, I only did very occasionally such outdoor work as is described, but I found it fairly often possible to encourage outdoor work without leaving my desk. Outdoor work can be set to teams of two or three for homework. One problem, in particular, I used to find very handy if not used too often. During much of my time at Charterhouse, my classroom was in a block which backed on one of the playing fields. There was in front of my window a nice little flowering tree,

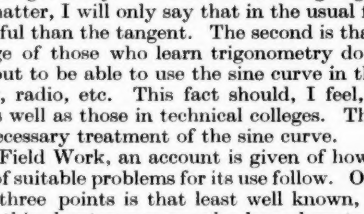


Fig. 2.

T, and at the back a convenient goalpost, *G*. The problem was to find the

direct distance between them. There was plenty of room for manoeuvre to the right, so by putting a stick at S the triangle TSG could be solved. I had some little pocket sextants, called "anglemeters", which I could lend for measuring the angle TSG, but I often insisted on its calculation first, from lengths only, and then the calculation of TG. The result had to be checked by some other method, and if the use of Pythagoras' theorem was suggested, the fact that 12 ft. of string with knots at 3 ft. and 7 ft. would be useful was pointed out. Finally, usually after no one had spotted it, and the calculations had been handed in, it could be pointed out that the midpoint theorem provided far the easiest method, and thus respect for this valuable result was greatly increased among my pupils.

One of the main features of the second part of the *Report* is Chap. XI on "Vectors and Complex Numbers". The production of this chapter was troublesome, and I hope that in my account of it the too frequent use of the first person singular will be pardoned.

The subject-matter of Chap. XI had always worried as well as interested me a good deal. Euler's use of $\sqrt{(-1)}$ or i seemed simple. There was merely the rule "$i^2 = -1$, but except for that go ahead as before". This, however, was too easy. Complex numbers had to be explained, and the first explanation came from the Argand diagram and the use of i as an operator. In some books, for example, Nunn's *Teaching of Algebra and Trigonometry*, there is a lot about i as an operator, and this seemed to settle the whole thing. However, when I bought a different book, there was nothing at all in it about i as an operator, but complex numbers became ordered pairs. This, again, seemed all right for addition, but the detestable rule for multiplication pretended not to come from Euler's manipulation of i, though I fear that it looked very much as if it did. Again, in either of these ways, complex numbers were interpreted as vectors, and a book on vector analysis showed that vectors were *added* exactly like complex numbers, but no explanation was given why they were not *multiplied* like complex numbers, but had to have two products, of which one was non-commutative. Finally, it was admittedly the case that the non-commutative business in vector products was started by Hamilton with his quaternions, but those who loved vectors seemed to detest quaternions, and never mentioned them, so that another tome had to be bought or borrowed to find out what was meant by the product of two quaternions.

So, thinking that an account of all these things ought to be made available in a single chapter, with a great deal more daring than discretion, I wrote out a first draft of such an account. Now I know nothing about the calculus of operations, and am only a recent and still somewhat suspicious convert to the doctrine of ordered pairs. Further, in teaching vectors I was for a long time among the "one chapter ahead" group of teachers, and finally I was and am almost completely ignorant of quaternions. So will you be surprised to hear that my script was not regarded as a success? A certain kindness to the Chairman is always probable in a committee, but even so the words "rubbish" and "poppycock" seemed apt to occur in the discussion of my script. However, I stuck to my view that our *Report* ought to include a comprehensive account of these matters, and finally this was agreed to. So we called in the experts. Professors Neville, Goodstein and Broadbent made up a team whose combined effort few will venture to call "poppycock", and these three, of course aided and abetted by the rest of us, re-wrote the chapter.

So now you have in fourteen pages a connected account of one of the most intriguing parts of mathematics, which previously would have needed quite a fair-sized library of mathematical books from which it could with difficulty

have been pieced together. I hope that even those teachers for whose pupils the first part of the *Report* is sufficient will themselves take the opportunity of reading Chap. XI.

I have just said that criticism of our team of experts is dangerous, yet even they need one caution. The teacher who says " it is easily shown " is a rash man, and in particular should beware of misprints. On the last line of p. 84, in almost the next word to " is easily shown ", what should have been a q has been somehow replaced by a p. So what could have been shown, perhaps easily, with a q, happens not to be true with a p. The moral, of which in conclusion I make a present to all, professors or not, is this: be chary of using the phrase " is easily shown ", but if you are rash enough to use it, mind your p's and q's.

Mr. G. L. Parsons (Merchant Taylors'): In speaking today about the teaching of trigonometry, with special reference to the recent *Report* of the Association, which, after many vicissitudes, has at last reached the members (incidentally providing for a large number of members who have joined since 1938 their first introduction to a very valuable side of the Association's work), I wish to confine myself almost entirely to one topic, namely, the close association between geometry and trigonometry. In doing so I do not wish to be thought to be detracting from the general usefulness of the *Report*, nor from those humorous and valuable points raised by the opener of this discussion, nor indeed from those which later speakers will doubtless add. But I do so because I feel strongly that, whatever the prestige and authority of this *Trigonometry Report* may be in the changing years that lie ahead, one characteristic by which the immediate post-war period will be remembered in regard to mathematical teaching is the increasing attention paid to the association between trigonometry and geometry.

It is indeed true, as stated in the *Report*, that some such idea has been put forward in a tentative way in the 1923 *Geometry Report* (p. 60). It is interesting to note, however, that the suggestion made in the 1923 *Report* is for a purely temporary liaison, with no thought of marriage at the end of it. Indeed it is quite confidently stated that the two subjects will subsequently separate. The amount of association, too, was to be strictly limited, and it was bounded by the somewhat cryptic condition that immediate re-interpretion into the language of similar triangles must always be possible. The compilers of the 1923 *Report* were, it seems, somewhat afraid of being stampeded, for they are careful to suggest that geometry examiners should sometimes set questions which expressly prohibit the use of trigonometry. In fact, such suggestions of an alliance between geometry and trigonometry as exist in the previous *Report* are very tentative and restricted in scope.

The *Trigonometry Report* attributes the comparative slowness with which this hint was developed to the fact that teachers were not prepared for it. I am inclined to think that the real reason lies elsewhere. Some time in the late twenties or early thirties there was an agitation against the complexity of the arithmetic papers set by one of the more popular examining bodies (I use the word " popular " solely in regard to the number of candidates). The examiners—or maybe it was the examining body—rejoined that the arithmetic papers had to be hard because many candidates got high marks for arithmetic (a defence which certainly could not be made nowadays), and this upset the balance of the subject. There was much discussion on this and similar topics. I remember particularly a meeting of the London Branch where many of the prominent members of that day debated with some vigour the question whether the introduction of trigonometry as an alternative to arithmetic would not prove to be a " soft option ", leading to deterioration in arithmetical work. Whether they were right about the " soft option " or not,

the gloomy prognostications of some of those present with regard to a decay in arithmetical accuracy have been all too amply fulfilled ; but perhaps that is a matter for another discussion. Be that as it may, it has always seemed to me that the step taken at that time, which suggested that trigonometry was the thing you naturally went on to if your class became saturated with arithmetic, was a false one. For it tended to suggest an emphasis on what is to my mind the wrong aspect of trigonometry. Instead of being semi-geometrical it became semi-arithmetical, and a term "Numerical Trigonometry" was invented to describe the sort of trigonometry you had to learn if you wanted to avoid the harder arithmetic questions in examinations.

There was a further complication which had to be taken into account and, though it sounds trivial, I think its effect upon the situation was not negligible. There were in some of these examinations children who offered arithmetic only. As these candidates also had the option of doing some trigonometry instead of the harder arithmetic, the questions set for trigonometry had to be carefully chosen in order to avoid mentioning any figure (for example, the circumcircle of a triangle) which might be unfamiliar to pupils who had "done" little or no geometry. This, in my opinion, tended to strengthen the false alliance between trigonometry and arithmetic to which reference has already been made.

The *Trigonometry Report* attributes the change of attitude in this matter to the *Report of the Conference of Educational Associations* in 1943. It is also stated that as a result of the syllabuses put forward in this *Report* (sometimes known as the *Jeffery Report*) various examining bodies introduced alternative syllabuses in which a greater fusion of geometry and trigonometry was encouraged. This statement as it stands is not quite correct ; in fact, in one important respect it puts the cart before the horse. For a modified alternative geometry-cum-trigonometry syllabus had already come into being in one of the well-known examining bodies. This syllabus was not in any way a direct product of this Association, though the framers of it were all members. It had—or perhaps I should say, it has (for it has still been retained by the body concerned)—some peculiarities which do not concern us here, but which may well lead to other debates in the future. The real point is that this syllabus, which already had introduced a considerable degree of fusion between geometry and trigonometry, was the primary cause of the Conference which led to the *Jeffery Report*.

I have spent some time on this past history, partly because I think the facts ought to be on record, and also because the history of the question considerably affects us. For I feel that this fusion of geometry and trigonometry is a most important development in the teaching of elementary mathematics, and perhaps deserves more stress in the *Report* than it actually receives. Much of the relevant material is there, in many cases nestling coyly under the title of "Miscellaneous Topics", but it is not quite brought together in the manner one might like to see. Trigonometry is quoted again and again as being a *unifying* factor. This it indeed is ; but it is also equally clearly an amplifying and explanatory factor filling up the gaps in geometry. These gaps are not in any way *inherent* in the study of space. They exist in the main because many of the classical geometrical results were enunciated at a period in the world's history when the technique of counting and calculating had not advanced far enough to allow the more numerical side of space relationship to be explored. By the time that the technique of calculation had reached this stage, geometry had become a "closed game" with fixed rules forbidding the introduction of numerical concepts. An echo of this attitude can be found in the statement in the 1923 *Report* that the proof of a certain theorem by Euclid's method is aesthetic and beautiful, whereas another

proof using trigonometrical ratios is merely effective. This is nothing more than a cleverly concealed enforcement of the " rules of the game ".

Let us consider one of the simplest facts in geometry. If a number of pupils are issued with certain measurements for the parts of a triangle usually known as b, c, A, we are generally convinced that the values they would get for a ought to be equal. Very nice—but equal to what, in relation to the original data? To this perfectly obvious and logical question geometry provides no answer, and it is only when you learn trigonometry or its equivalent that you arrive at the fact that

$$a^2 = b^2 + c^2 - 2bc \cos A$$

and settle the question. No circular argument is involved : I say this because " circular argument " used to be the bogey man with which all would-be users of trigonometry in geometrical questions were threatened. For the same notions of similarity which permit you to invent $\cos A$ also permit you to invent Pythagoras' theorem : in fact, one of the easiest ways of writing out the proof of Pythagoras' theorem is to write down two values of $\cos A$ and equate them, and then to do the same for $\cos B$. As the *Trigonometry Report* rightly points out, once you have reached the cosine rule the formidable difficulties of the geometrical statement of the " extension of Pythagoras " have all but disappeared, and there are other geometrical situations where similar considerations apply. I suppose that there is not a single teacher of mathematics who has not experienced the slight feeling of exasperation and frustration that I always feel at the purely geometrical statement that triangles which are equiangular have their sides in the same proportion. This is an admirable thing to know—but what *is* the proportion? One can only feel sympathy with the pupil who assumes that, since you have not told him what the proportion is, it must be what appears to him the reasonable and obvious one—the ratio of the measures of the angles—and also with his sense of irritation and hostility towards the teacher who thereupon condescendingly points out that this assumption has the undesirable effect of reducing his 60° set-square to a straight line.

The effect of this relationship between the subjects is a two-way traffic. If trigonometry amplifies and consolidates geometry, geometry in its turn enlightens and elucidates trigonometry. It is useful and practical to know that

$$a/\sin A = b/\sin B = c/\sin C :$$

it is only when a geometrical transformation of the figure adds the fact that each expression is equal to $2R$ that the practical formula becomes a piece of mathematical knowledge. When the geometrical facts about the areas of triangles and parallelograms have been stated in a trigonometrical way in terms of $ab \sin C$, geometry can soon repay its debt to trigonometry, for in considering the area of a parallelogram in two ways the pupil is easily led to the conviction that no other value for $\sin (180° - C)$ except $\sin C$ is possible, and that if the sine of an obtuse angle may not be taken as equal to the sine of its supplement, then it is quite useless to consider trigonometry in relation to obtuse angles.

Similar considerations obviously apply to the cosine (*via* the extension of Pythagoras) and to the formulae for $\sin 2A$, $\sin (A + B)$, etc. One such example, which is not so frequently mentioned, is the geometrical derivation of those extremely useful formulae $\sin \theta = 2t/(1 + t^2)$, $\cos \theta = (1 - t^2)/(1 + t^2)$.

The beneficial influence of trigonometry in amplifying and clarifying geometry is by no means confined to two-dimensional geometry. The *Trigonometry Report* is perhaps a little disappointing in its references to three-dimensional work. For the only direct reference to three-dimensional work

seems at first reading to stress rather an opposite idea, that three-dimensional geometry is something you ought to introduce in order to make trigonometrical examples more interesting. Probably this is only another way of saying the same thing, and doubtless the authors would agree that the knowledge of the three fundamental ideas of solid geometry which the ordinary pupil might be expected to know—I mean the perpendicular to a plane, the angle between a line and a plane, and the angle between two planes—is much assisted by being presented against a trigonometrical background.

One final short remark concerning the teaching of trigonometry : it provides one of the instances—perhaps the clearest and most striking of the instances—in which the ordinary pupil can make an acquaintance with the evolutionary character of mathematics. I refer, of course, to the way in which an easy fundamental concept such as the sine of an angle is extended by mathematical thought to include cases not covered by the original definition. This is a characteristic of mathematical thought. To appreciate it is surely part of one's education, and nowhere is it met more clearly than in the extension of the definition of trigonometrical ratios first to obtuse angles, and later to angles of any magnitude.

Mr. A. W. Siddons (formerly Harrow) urged that children should be able to write down the sine and cosine definitions for a triangle in any position. He deplored the use of mnemonics, and considered that the diagram on p. 62 of the *Report*,

Sin	All
Tan	Cos

should not be used ; an appeal to first principles was preferable.

Dr. E. A. Maxwell (Queen's College, Cambridge) pointed out that a common formulation of the proof of the addition formulae by means of the cosine rule was not valid for angles greater than 180°.

Mr. A. Geary (Northampton Polytechnic) complained that the *Report* was not clear in its definitions of vectors and complex numbers.

Mr. W. J. Langford (Battersea Grammar School) asked whether members approved of the space given to tabulating methods for the solution of triangles.

Mr. F. J. Tongue (Kingswood) approved of the emphasis on methods of tabulation, and spoke of the value of checks on numerical work. He pointed out that the figure for sin $(A+B)$ on p. 63 of the *Report* could also be used for cos $(A+B)$.

Mr. R. Smart (Heriot-Watt College, Edinburgh) approved of the tabulations, and referred to details given in McRobert and Arthur, *Trigonometry*. He also emphasised the value of checks.

Miss F. M. Holgate (Manchester) suggested that to begin with the tangent was a practical approach for the younger children.

Miss H. M. Cook (Edge Hill Training College) spoke of the value of simple surveying.

Mr. A. Robson (formerly Marlborough) reminded members that Reports should not dwell on work contained in textbooks, save when the treatment given in textbooks needs correction. He preferred the half-angle formulae to the cosine formula in solving triangles.

Mr. C. E. Kemp (Reading) spoke of the value of the tabulation of methods of solving triangles as a collection of such schemes.

Professor E. H. Neville (Reading) emphasised that a Report of the Association is not a textbook.

THE GEOMETRY REPORTS.

By C. O. Tuckey.

The recently-published *Trigonometry Report* makes incidentally a good many suggestions bearing on the teaching of geometry, and the time has perhaps come to consider to what extent the *First* and *Second Geometry Reports* continue to give useful guidance to young teachers and whether any parts of them are nowadays out of date. It may be that one who has been intimately associated with both these Reports should make an attempt to do this.*

During this half-century there has been a continuous movement in the teaching of geometry away from the formal setting-out of a series of propositions in logical sequence as in Euclid's *Elements* and towards the somewhat haphazard discussion of various geometrical properties arrived at by methods which permit the use of algebra, trigonometry and perhaps accurate drawing. This change may also be described by saying that at the start of the century a pupil was expected to learn the proofs of 100 theorems ; about 1925 teachers were content to exact proofs of about 50 theorems, while in 1951 they consider themselves fortunate if their pupils master a dozen. Again in 1900 the geometry teaching was throughout in the same stage, that of euclidean proof ; in 1925 three stages, *A* the experimental, *B* the deductive, and *C* the systematising stage, were beginning to be recognised ; while in 1951 it rather looks as if few pupils ever get beyond stage *B*.

Now the *First Geometry Report*, published in 1923, though it was not the first suggestion of Stages, was highly important in persuading teachers to adopt the idea that there should be stages in the *method* of teaching (not merely in the subject-matter), and that the strict logic suitable to the latest times of school teaching should not be forced on beginners. The battle in this matter having been won and many of the suggestions in the Report for the " organisation of the primitive and of the derived propositions " having been re-considered, restated and in some cases reversed in the *Second Geometry Report*, the young teacher may reasonably ask whether this *First Report* has any value for him or whether it is only of historical interest. A partial answer to this question is given on the first page of the *Second Geometry Report* :

" There are one or two minor differences, but on the whole the teacher who reads Chapters I and II of the older report *pari passu* with Chapters II to IX of this one, will find that what is said here elucidates and amplifies them. . . . The main bulk (about two-thirds) of the 1923 Report, namely Chapters III to VII, is concerned with questions which are not dealt with at all in the present one. . . ."

If this quotation represented a correct view in 1938, what about these Chapters III to VII when looked at in 1951?

Chapter III, " Discussion of Some Disputed Points," deals with eight items, and of these all except (3) and (7) and possibly (1) are as valuable now as when they were written. Item (3), on the possibility of replacing the " parallel postulate " by a " principle of similarity ", always to be associated in the minds of those who knew him with the late Sir Percy Nunn, is perhaps now only of academic interest, and the textbook by Nunn and a collaborator which was to persuade the world how simple this replacement could be made to seem, though written, will never be published.

Geometry nowadays starts by assuming, as made obvious by drawing, all the angle properties of parallel lines, and the existence of similar figures, and

* Both Reports are in print, and copies can be obtained from Messrs. Bell, Kingsway, W.C. 2.

in so doing has swallowed at a gulp both the rivals mentioned. It is only those with leisure and inclination to " chew the cud " that are likely to give the matter serious consideration.

Chapter IV, on " The Difficulties of an Agreed Sequence ", is out of date ; when there is no sequence there is no need to consider the difficulties which would arise if there were one.

Chapter V, " Notes on Some Minor Points," states one which has proved of great importance, the suggestion that a group of theorems should be arranged as a bunch of bananas rather than a string of sausages. The chapter has not lost its value, nor has Chapter VII, " A Note on Relativity," but Chapter VI, " A Word with the Examiner," might well be regarded as out of date.

To sum up, about half the 1923 Report is in 1951 of value to the teacher, and it is only fair to add that the other half, most of whose wording was contributed by Messrs. Nunn and Neville, is worth reading for those who have a liking for the occasional striking phrase or *mot juste*.

To turn to the *Second Geometry Report* : this is a much more detailed and some would say a much more humdrum document. The aim of a great part of the text was to give, in some detail, suggestions for class-room teaching, in the hope that the reader might say to himself : " That seems an excellent idea. I must try it on the Upper Fourth." Turning over the pages it may seem to the present-day teacher that there is far too much about " theorems ", and that since only a dozen or so of " theorems " are studied nowadays, the Report must be out of date. This is a misunderstanding, due to the spread of the detestable and vulgar habit of restricting the use of the word " theorem " to those labelled in the textbook Theorem I, Theorem II, etc., whose proofs must be " mugged up " before an examination can be faced. A theorem is some property to be proved, and is not less a theorem if the proof has not to be " mugged up ", nor even if the proof has never been found, as in the cases of Fermat's Last Theorem or the Four-Colour Theorem. The theorems to be " mugged up " are in this second Report described as " standard theorems ", and the fact that there were a good many more of these included then in the elementary course than is the fashion nowadays does not deprive of its value the discussion in Chapter V of the " Groups of Theorems "—to be called " Groups of Properties " nowadays—or of the following chapters on Stage *B* teaching.

One important matter should, however, be mentioned. The discussion on pp. 80–1 of Standard Theorems was directed to the method of teaching customary when, as stated earlier, the aim was that the proofs of about fifty standard theorems should be known. This was a view about which members of the committee were by no means in agreement, and the view of the minority was set out in Appendix 14, referred to on p. 81. This minority was headed by W. C. Fletcher, formerly chief inspector of secondary schools, and probably the man of greatest ability on the committee, and the ideas set out in Appendix 14 on the " Relative Importance of Propositions and Riders " proved the most influential in the whole Report, and led directly to the geometrical part of the modern " alternative syllabus " crystallised later in the *Jeffery Report*.

Another point should be mentioned. The Mathematical Association has for these fifty years been urging the importance of solid geometry. It did so in the 1923 Report with little success, and returned to the attack in 1938, when both simple and more advanced explanations for the drawing of diagrams of solids are included in Appendices 3, 4 and 5, and many riders on solid figures are suggested. Some progress has no doubt been made here, but it is difficult to be satisfied with it.

To sum up. The *Second Geometry Report*, except for a few sections here and there, is not out of date, but is full of ideas likely to be valuable for the

young teacher. He, however, should supplement his study of this Report by a careful study of the *Jeffery Report*, on which the syllabus for the ordinary level of the General Certificate of Education is likely to be based for some time.

It remains perhaps to consider whether the time has come for a third geometry report. Such a report would refer to and supplement the *Second Report*, but would incorporate, perhaps by direct quotation, all that was considered still to be valuable in the *First Report*, so that the young teacher might be expected to study not the first and second reports as at present, but the second and third. The main content of a third report would be a discussion of how to teach to the Jeffery syllabus. This involves a difficulty which perhaps there is not yet sufficient experience to solve. Consider the sentences: " Proofs of the key-theorems and deductions of the others form an important part of the geometry teaching, but it is doubtful whether any proofs of theorems should be required in examinations. If all the key-theorems and derived theorems are liable to occur, *too much teaching time will be sacrificed to their preparation.* The theorem work in examinations might well be limited to the key theorems." (The italics are not in the original.) In present-day examination syllabuses, the last sentence seems to have been or to be likely to be adopted as a working rule. Thus we escape from the *lamentable* state of affairs when the foolishly over-zealous teacher acted up to the phrase italicised above by cramming his pupils with the proofs of all the properties of which " a sound appreciation will be expected ". But we are landed in the illogical and indeed *ridiculous* position in which an examiner may ask the candidate to prove the product property of the circle for an external point, but may not ask him to extend the identical argument to prove it also for an internal point.

The position, however, is not merely illogical. It has its inherent dangers. The " deductions of the other properties " from the key-theorems not only " forms an important part of the geometry teaching ", but is bound to take up a considerable fraction of the available time. At first the teachers, trained on these deductions and accustomed to teach them, will no doubt continue to do so. But it is a good deal to ask that anything from a third to a half of the time given to geometry should be devoted to stuff that cannot be used in examination. The " sound appreciation " will tend in the classes of teachers pressed for time to become hasty verification by drawing and perhaps just rote-learning. The phrase " it is obvious " may even be adopted by the teacher from his pupils. If this takes place, the reform of geometrical teaching will have gone too far. It will be for those who prepare a third report on geometry to consider the dilemma suggested above and to find a way out of the difficulty.

C. O. T.

GLEANINGS FAR AND NEAR.

1681. What a brain! That is what we all said when we visited the National Physical Laboratory at Teddington to-day, to see Britain's latest computing " electronic brain " in action.

Now take a " simple " problem like 3,971,428,732 multiplied by 8,167,292,438. If you are in the skilled mathematician class it would take you eight minutes to work out that little problem. The brain which cost £40,000 takes one 500th of a second.—*Evening News*, November 29, 1950. [Per Mr. H. V. Lowry.]

MATHEMATICAL NOTES.

2229. *Acute or obtuse?*

Looking again at Note 1991 (October, 1947), I found myself echoing its closing words, in the question : why is this not a familiar elementary proposition? A question which answered itself in terms of the undue difficulty that seems to have been felt about the separation of points in the xy-plane by the line (a, b, c). This difficulty I had discussed in XXX, 296, pp. 203–4 (October, 1946) ; and I find that the procedure set out there leads at once to elementary treatment of the property here in question.

On the hypothesis, we may write

$$aS = (ax + \beta_1 y + \gamma_1)(ax + \beta_2 y + \gamma_2) = L_1 \,.\, L_2, \text{ say ;}$$

and we may obviously suppose a positive, without loss of generality (or, alternatively, b positive—unless $a = 0 = b$, the excluded case for a rectangular frame).

(i) Since $\beta_1\beta_2 = ab$, β_1 and β_2 are of the same sign if b (as well as a) is positive ; and the two lines have then *gradients of the same sign.* The facts are then as in one or other of the diagrams of Fig. 1 ;* and it is evident that $S \equiv L_1L_2$ is *negative* for points in the *acute* angle of the line-pair.

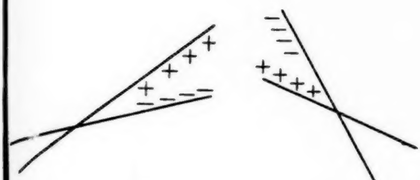


FIG. 1

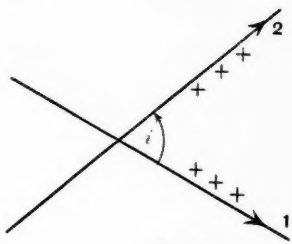


FIG. 2

(ii) But if b is negative, β_1, β_2 are of opposite signs and the case is as in Fig. 2 : so that it is necessary to discuss whether the angle i is *acute* or *obtuse* (or *right*).

Again, there is obviously no loss of generality in supposing β_1 positive, β_2 negative (as in the figure) ; and $i = \psi_2 - \psi_1$, where ψ_1, ψ_2 specify acute inclinations (negative and positive, respectively) to Ox, and $\tan \psi = -a/b$.

Hence
$$\tan i = a(\beta_2 - \beta_1)/(a^2 + \beta_1\beta_2)$$
$$= -(\beta_1 - \beta_2)/(a + b),$$

and $\beta_1 - \beta_2$ is (by hypothesis) positive. It follows that the i-angle is acute or obtuse according as $(a + b)$ is negative or positive; thus $(a + b) \,.\, S \equiv (a + b) \,.\, L_1 \,.\, L_2$ is, *in either case*, negative for points in the acute angle of the line-pair. ($a + b = 0$ is, of course, the excluded case.)

(iii) Hence the general theorem (for a rectangular frame).†

(iv) For a non-rectangular frame, this is just the type of case to which transformation to (the most simply related) rectangular coordinates is pecu-

* L positive for points to the same side of the line as $(+\infty, 0)$, a being positive.

† That the discriminating expression is homogeneous of the second degree *in the coefficients* is, of course, significant to the generality of the result.

liarly applicable—because of the trigonometry.

Using $\xi = x - \cos\omega \,.\, y, \quad \eta = \sin\omega \,.\, y,$

whence $\sin\omega \,.\, x = \sin\omega \,.\, \xi + \cos\omega \,.\, \eta, \quad \sin\omega \,.\, y = \eta,$

$$\sin^2\omega \,.\, S = a\sin^2\omega \,.\, \xi^2 + \ldots + (a\cos^2\omega - 2h\cos\omega + b)\eta^2 + \ldots$$

Hence the criterion by the sign of $(a - 2h\cos\omega + b) \,.\, S$.

The purpose of this Note is to stress the importance and the usefulness of the discrimination by sign of the points (of the xy-plane) to either side of "*the line* (a, b, c)"—in terms of *that given specification*. D. K. PICKEN.

2230. *Au sujet de certains polygones convexes inscriptibles.*

Théorème. Dans un polygone convexe $(P) \equiv A_1A_2 \ldots A_{4n+2}A_1$ *de* $2 \,.\, (2n+1)$ *côtés, inscrit à une circonférence* (O)*, si le produit des côtés de rang pair est égal à celui des côtés de rang impair, et si* $2n$ *de ses diagonales principales se rencontrent en un même point, les* $2n+1$ *diagonales principales sont concourantes et réciproquement.*

En effet, si les $2n$ diagonales $A_1A_{2n+2}, \ldots, A_{2n}A_{4n+1}$ concourent en un point M, la droite $A_{2n+1}M$ recoupe la circonférence (O) en un point B situé nécessairement entre A_{4n+1} et A_1.

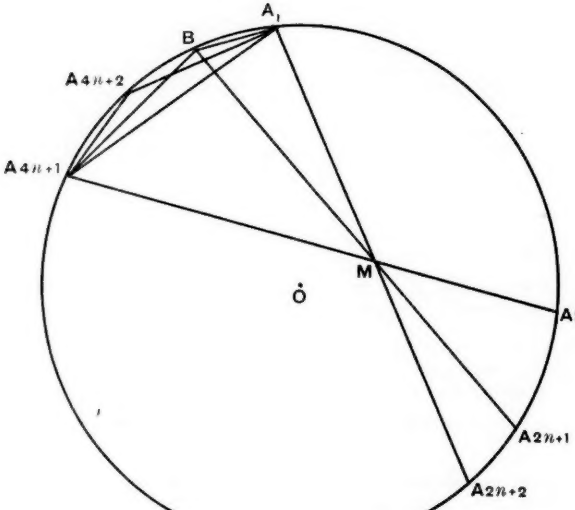


FIG.

D'après un théorème connu, qui constitue la réciproque de celui qu'il s'agit de démontrer, le produit des côtés de rang pair du polygone de $2 \,.\, (2n+1)$ côtés $A_1A_2 \ldots A_{4n+1}BA_1$, inscrit à la circonférence (O), est égal à celui des côtés de rang impair, ce qui donne

$$A_2A_3 \,.\, A_4A_5 \,.\, \ldots \,.\, A_{4n+1}B = A_1A_2 \,.\, A_3A_4 \,.\, \ldots \,.\, BA_1. \quad \ldots\ldots\ldots\ldots\ldots\text{(i)}$$

Or, comme on a, par hypothèse,

$$A_2A_3 \,.\, A_4A_5 \,.\, \dots \,.\, A_{4n+1}A_{4n+2} = A_1A_2 \,.\, A_3A_4 \,.\, \dots \,.\, A_{4n+2}A_1, \quad \text{.........(ii)}$$

il résulte de ces égalités que

$$A_{4n+1}A_{4n+2} : A_{4n+1}B = A_{4n+2}A_1 : BA_1 \text{.......................(iii)}$$

Les triangles $A_1A_{4n+2}A_{4n+1}$ et A_1BA_{4n+1} ayant un angle égal en B compris entre deux côtés homologues proportionnels sont donc semblables et les angles $(A_1A_{4n+2}, A_1A_{4n+1})$ et (A_1B, A_1A_{4n+1}) sont égaux ; les points A_{4n+2} et B se confondent et toutes les diagonales principales du polygone (P) concourent au point M.

N.B. Le cas particulier où $n = 1$ a été signalé par Mathot.*

V. THÉBAULT.

2231. *Sur des cercles associés à un triangle.*

Dans la note 1799 (*Mathematical Gazette*, 1945, p. 18) Dr. B. E. Lawrence a établi une propriété des cordes interceptées sur les côtés d'un triangle par la circonférence qui passe par les pieds de trois bissectrices concourantes, qui est un cas particulier de la proposition suivante :

Théorème. La circonférence qui passe par les pieds des céviennes AL, BM, CN qui divisent les côtés BC, CA, AB d'un triangle ABC dans les rapports

$$BL/LC = c^n/b^n, \quad CM/MA = a^n/c^n, \quad AN/NB = b^n/a^n,$$

n étant un exposant arbitraire, intercepte sur les droites BC, CA, AB des segments dont les mesures x, y, z satisfont à la relation, en grandeur et en signe,

$$x/a^{n-1} + y/b^{n-1} + z/c^{n-1} = 0, \text{..............................(i)}$$

a, b, c désignant les mesures des côtés BC, CA, AB du triangle.

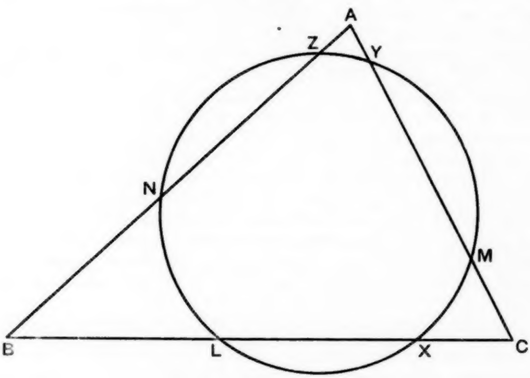


FIG.

En effet, soient AL, BM, CN les céviennes en question et les cordes LX, MY, NZ de longueurs x, y, z. Nous avons

$$AM \,.\, AY = AN \,.\, AZ,$$

* Mathesis, 1897, 159.

R

et ainsi de suite, d'où les èquations relatives à la figure

$$\frac{bc^n}{(c^n+a^n)^2}\{bc^n-(c^n+a^n)y\}=\frac{cb^n}{(a^n+b^n)^2}\{cb^n-(a^n+b^n)z\},$$

$$\frac{ca^n}{(b^n+c^n)^2}\{ca^n+(a^n+b^n)z\}=\frac{ac^n}{(b^n+c^n)^2}\{ac^n+(b^n+c^n)x\},$$

$$\frac{ab^n}{(b^n+c^n)^2}\{ab^n-(b^n+c^n)x\}=\frac{ba^n}{(c^n+a^n)^2}\{ba^n+(c^n+a^n)y\}.$$

En multipliant ces équations respectivement par a^{2n}, b^{2n}, c^{2n} et en additionnant, on obtient la relation (i).

$$\text{Si } n=1, \quad x+y+z=0\,; \qquad \text{(ii)}$$

c'est la relation de Lawrence.

Lorsque $n=2$, les céviennes coïncident avec les *symédianes* du triangle ABC, et

$$x/a+y/b+z/c=0.$$

Corollaire. La circonférence qui passe par les isotomiques des pieds des céviennes AL, BM, CN intercepte sur les côtés du triangle ABC des segments dont les mesures sont liées par la relation, en grandeur et en signe,

$$a^{n+1}x+b^{n+1}y+c^{n+1}z=0.$$

Elle s'obtient de la même manière que la relation (i) et, si $n=-1$, on retrouve (ii).

V. Thébault.

2232. *The equations for the centre of a conic.*

The usual excellent method for finding the centre of the general conic by means of the r-quadratic often leaves in the pupil's mind the feeling that it is a fluke that the answer should involve partial derivatives.

The following method, which brings them in directly, is probably neglected because it assumes too much, but might be considered as a possible alternative.

If the conic is $f(x, y)=0$, then $f(x, y)$ is a function which changes sign as we cross the curve. Suppose $f(x, y)$ written so as to be negative inside the curve, for example, as $x^2/a^2+y^2/b^2-1$, not as $1-x^2/a^2-y^2/b^2$. Then if PQ is a chord parallel to the x-axis, $f(x, y)$ is zero at P, decreases to a minimum as we pass along PQ, and is zero again at Q. The rate of change of $f(x, y)$ is $\partial f/\partial x$, and the minimum occurs when $\partial f/\partial x=0$. But for a quadratic function the minimum occurs half-way between the zeros, and so $f(x, y)$ is minimum at the midpoint of the chord. Thus $\partial f/\partial x=0$ is the locus of midpoints of chords parallel to the x-axis, and it is a straight line, being of the first degree. Similarly $\partial f/\partial y=0$ is a straight line which is the locus of midpoints of chords parallel to the y-axis. But the centre is the only point where two chords in different directions are bisected :

hence the equations $\partial f/\partial x=0$, $\partial f/\partial y=0$ give the centre.

Caunt, *Infinitesimal Calculus*, pp. 491–2, proves that $\partial f/\partial x=0$ is the line joining the points of contact of tangents parallel to the x-axis, and similarly for $\partial f/\partial y=0$; thus these lines meet at the centre.

C. O. Tuckey.

2233. *A symmetrical figure to demonstrate Pythagoras' Theorem.*

An appeal to symmetry in presenting geometrical facts is always effective, but unfortunately most figures are obstinately unsymmetrical. The well-known tile pattern does not really take us far on the road with Pythagoras, and though demonstrations of his theorem are many, I have not seen one which

makes use of symmetry while at the same time embodying the basic figure of the right-angled triangle with the squares described on its sides. Here, however, is one which I have used with pleasing results. With the help of a little shading and a brief legend the import of the figure is quickly grasped by pupils of moderate or mediocre ability.

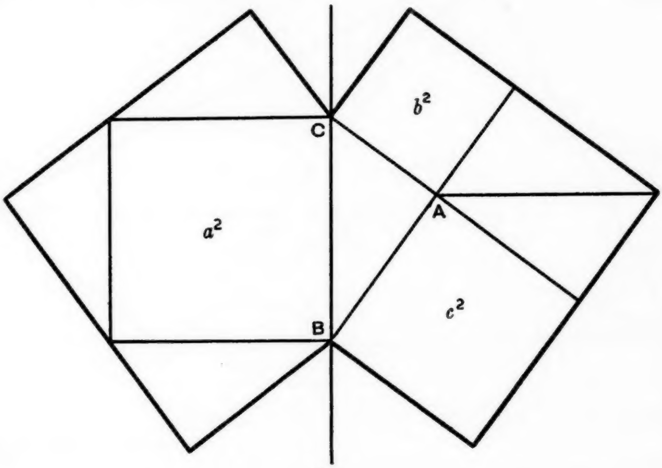


FIG.

The figure can conveniently be drawn on a $8'' \times 6''$ sheet of squared paper, using the triangle whose sides are 2·5, 2·0 and 1·5 inches, and whose altitude, length 1·2 inches, divides the hypotenuse into segments 0·9 and 1·6 inches. It is as well to arrange that the horizontal diagonal of the rectangle falls on one of the more heavily printed " inch " lines. A. G. SILLITTO.

2234. *An analogue of Ceva's Theorem.*

In Ceva's Theorem for the triangle we can regard the ratios of lengths as number pairs. Thus $AB : BC$ may be written (AB, BC). We make the convention that the number pair (a, b) is identical with the number pair (ka, kb). Multiplication is defined by the equation

$$(a, b) \cdot (c, d) = (ac, bd).$$

Number Triples. In the same way we can regard the number triple (a, b, c) as the ratio $a : b : c$. We make the convention that the triple (a, b, c) is equal to (ka, kb, kc) and define multiplication by the equation

$$(a, b, c) \cdot (d, e, f) = (ad, be, cf)$$

Theorem. If points E, F, K, L be taken on the faces (produced if necessary) BCD, ACD, ABD, ABC respectively of the tetrahedron $ABCD$, in such a manner that AE, BF, CK, DL are concurrent then

$$(BCE, CDE, DBE) \cdot (ADF, ACF, CDF) \cdot (ADK, BDK, ABK) \cdot (BCL, ABL, ACL) = (1, 1, 1)$$

where ABC represents the area of triangle ABC.

Proof.

Let the four concurrent lines meet in O.

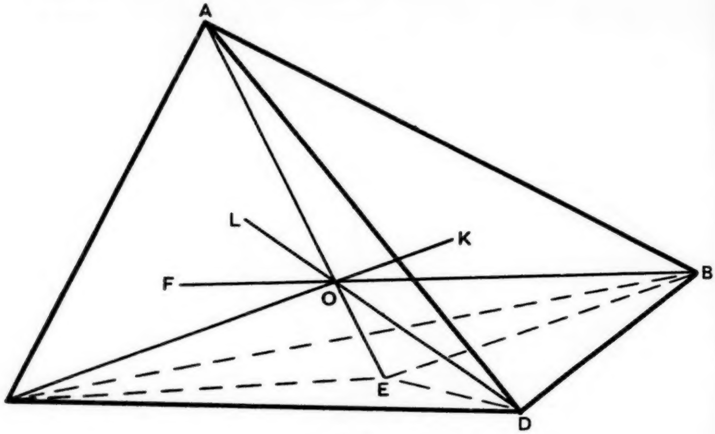


Fig.

Then $(BCE, CDE, DBE) = (OBCE, OCDE, ODBE)$

$$= (ABCE, ACDE, ADBE)$$

where $ABCD$ represents the volume of the tetrahedron $ABCD$.

Hence we have by subtraction

$$(BCE, CDE, DBE) = (OABC, OACD, OADB).$$

Similarly

$$(ADF, ACF, CDF) = (OADB, OABC, OBCD)$$

$$(ADK, BDK, ABK) = (OACD, OBCD, OABC)$$

$$(BCL, ABL, ACL) = (OBCD, OADB, OACD).$$

Hence the product is equal to $(c, c, c) = (1, 1, 1)$ (where c is equal to $OABC \,.\, OADB \,.\, OACD \,.\, OBCD$). ALAN ROSE.

2235. *Note on a problem in solid geometry.*

Certain questions of elementary geometry are best solved by purely geometrical methods while others are more suited to analytical treatment. It often happens that some questions are worth solving by both methods as each method clarifies certain aspects of the problem. This is illustrated by the following problem :

" If $ABCD$ is a tetrahedron and $A'B'C'D'$ its polar tetrahedron with respect to a quadric S, then in general AA', BB', CC', DD' are generators of a uniquely determined quadric S'. If $ABCD$ is self-conjugate, then the lines AA', BB', CC', DD' are concurrent."

A purely geometrical proof is sketched below. In order to prove that AA', BB', CC', DD' generate a quadric, in general it will be sufficient to show that they have more than two transversals. We project from A' on the plane BCD so that $B' \to B_1$, $C' \to C_1$ and $D' \to D_1$. The triangles BCD, $B_1C_1D_1$ are then polar triangles with respect to the conic in which the plane BCD meets the

quadric. They are thus in perspective from a point Z: and it is easy to show that the line $A'Z$ meets the four lines AA', BB', CC', DD'. Similarly we can find three other transversals through B', C' and D'. It follows that the lines AA' etc. generate a definite quadric S'.

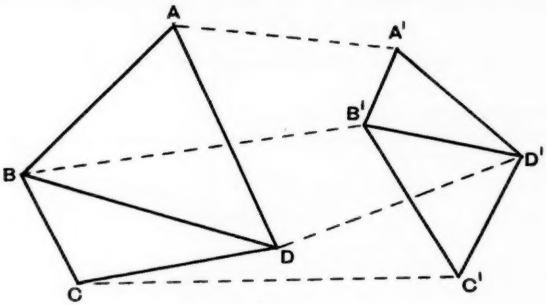


FIG.

To prove the second part, we see that if $ABCD$ is self-conjugate (*i.e.* AB meets $A'B'$, etc.) then the three points $(BC, B'C')$, $(AC, A'C')$, $(AB, A'B')$ are collinear as they lie in both the planes ABC, $A'B'C'$. The triangles ABC, $A'B'C'$ lying in different planes are in perspective, and AA', BB', CC' are concurrent at O. Similarly AA', BB', DD' are concurrent at O. This completes the proof of the second part.

When the tetrahedron $ABCD$ is self-conjugate the proof which we have given of the first part becomes invalid, since any line through O is a transversal to the four lines AA', BB', CC', DD'. In this case the quadric S' appears to degenerate into a cone. Suppose we start with a tetrahedron $ABCD$ which is not self-conjugate. We might imagine this tetrahedron to be deformed continuously in such a way that it finally becomes self-conjugate. It seems plausible that the quadrics S' corresponding to the various positions of the tetrahedron will determine, in the limiting position, a definite quadric cone. But since we then have only four generators and we require five to determine a quadric cone, the situation seems rather obscure.

Light is thrown on this particular aspect of the problem by the analytical solution outlined below.

Choose $ABCD$ as tetrahedron of reference, and for the quadric S the equation $\Sigma a_{rs}x_r x_s = 0$ where $|a_{rs}| \neq 0$. Then the point A' has coordinates $(A_{11}, A_{12}, A_{13}, A_{14})$, where A_{ij} is the cofactor of a_{ij} in the determinant $|a_{ij}|$.

The line AA' has equation $x_2/A_{12} = x_3/A_{13} = x_4/A_{14}$;

BB' has equation $x_1/A_{21} = x_3/A_{23} = x_4/A_{24}$;

and similarly for CC' and DD'.

It is easy to verify that the four lines AA', BB', CC', DD' all lie on the quadric S' whose equation is

$$(A_{24}x_1 - A_{21}x_4)(A_{13}x_2 - A_{12}x_3)(A_{14}A_{32} - A_{12}A_{34})$$
$$= (A_{23}x_1 - A_{21}x_3)(A_{14}x_2 - A_{12}x_4)(A_{24}A_{31} - A_{21}A_{34}).$$

The coefficient of x_1x_2 in this equation is $A_{12}A_{34}(A_{23}A_{14} - A_{24}A_{13})$: and by use of a well-known theorem of determinants, this is $A_{12}A_{34}|a_{ij}|(a_{13}a_{24} - a_{23}a_{14})$.

Now if $ABCD$ is self-conjugate, then $a_{13}a_{24}=a_{23}a_{14}$ etc., and the equation of the quadric S' becomes evanescent, and S' is thus indeterminate ; this corresponds to the fact that the four generators at O are insufficient to determine the quadric S'.

The conditions $a_{13}a_{24}=a_{23}a_{14}$ etc., imply

$$A_{23}/A_{24}=A_{13}/A_{14} \text{ etc.}$$

which is the condition for the concurrency of AA', BB', CC', and DD'.

T. J. WILLMORE.

2236. *Isogonal points.*

The proof given in most textbooks has always seemed to me unsatisfying and rather off the point.

Given $\angle PAB=\angle QAC, \quad \angle PBA=\angle QBC,$

to prove that $\angle PCA=\angle QCB.$

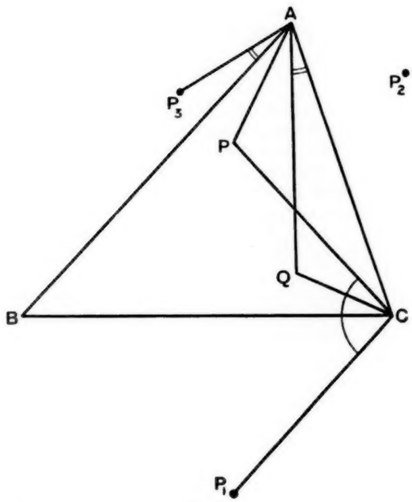


FIG.

Let P_1, P_2, P_3 be the images of P in the sides BC, CA, AB of the triangle.

Then $\angle QAP_3=\angle A=\angle QAP_2$

and $AP_3=AP_2$.

Hence Q lies on the perpendicular bisector of P_2P_3, and also on that of P_3P_1 ; that is, Q is the circumcentre of the triangle $P_1P_2P_3$. Hence

$$\angle QCP_1=\angle QCP_2=\angle C.$$

Thus $\angle QCB=\angle PCA.$

E. W. BURN.

2237. *The inequality between the arithmetic and geometric means.*

Let P denote the product, and S the sum of n positive numbers $a_1, a_2, \dots, a_n$.

The inequality

$$\sqrt[n]{P} \leqslant S/n \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

is true for $n = 2$. Let $a > 0$,

$$Q = (Pa)^{1/n(n+1)},$$

and suppose that (i) holds for some $n \geqslant 2$. Then three applications of (i) lead to

$$\begin{aligned} Q^n &= \{(Qa^{(n-1)/n})(P^{1/n})^{n-1}\}^{1/n} \\ &\leqslant \frac{1}{n}(Qa^{(n-1)/n}) + \frac{n-1}{n}P^{1/n} \\ &\leqslant \frac{1}{n}(Q^n a^{n-1})^{1/n} + \frac{n-1}{n^2}S \\ &\leqslant \frac{1}{n}\left(\frac{1}{n}Q^n + \frac{n-1}{n}a\right) + \frac{n-1}{n^2}S. \end{aligned}$$

Since $n - 1 > 0$, the extreme members give

$$\sqrt[n+1]{(Pa)} \leqslant (S + a)/(n + 1).$$

Thus (i) holds for $n + 1$ numbers, and hence generally. The proof also shows that there is equality in (i) if and only if $a_1 = a_2 = \ldots = a_n$. V. Popovic.

2238. *The swastika.*

The curve given by $x^4 - y^4 = 2xy$ has its graph as shown in Fig. 1 (*a*). If we invert with respect to the circle $x^2 + y^2 = 1$ its equation becomes

$$x^2 - y^2 = 2xy(x^2 + y^2).$$

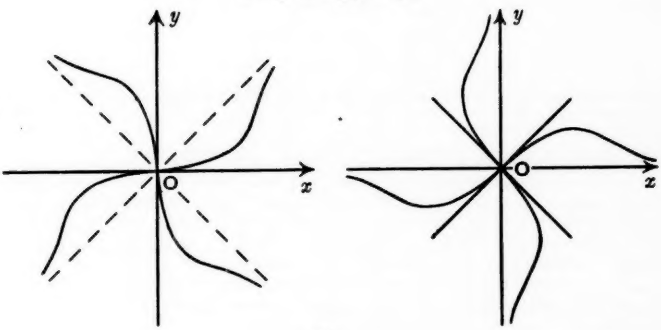


Fig.

and its graph is as in Fig. 1 (*b*). This is a curve similar to the previous one, with the asymptotes turned through an angle of $\frac{1}{4}\pi$, and the curve reflected in the asymptotes.

The curves are perhaps the nearest graphical approximation to Swastikas, and by a little mathematical symbolism may be supposed to show that totalitarian rule in two different countries, with different ideologies and with different points of view even to the extent that one is the inverse of the other, may produce in the long run the same pattern or effect. H. N. Haskell.

2239. *A simple construction for determinants of given value N.*

We shall deal with determinants $|a_{ij}|$, $(i, j = 1, 2, 3)$ whose elements in any two columns are positive integers, and whose elements in the third column are positive or negative integers or rationals, or zero.

Lemma.

If the sum of the elements in a row is N,

$$a_{i1}+a_{i2}+a_{i3}=N,$$

then it is seen by expansion that

$$\begin{vmatrix} a_{11} & a_{12} & \alpha_{13} \\ a_{21} & a_{22} & \alpha_{23} \\ a_{31} & a_{32} & \alpha_{33} \end{vmatrix} = N(A_{13}-A_{23}+A_{33}),$$

where $\alpha_{i3}=N-(a_{i1}+a_{i2})$ and A_{i3} is the minor of α_{i3}.

Construction.

The construction of a determinant for value N now follows easily. We first obtain elements for which

$$A_{13}=+1\,;\quad A_{23}=A_{33}=-1.$$

The Fibonacci series provides numbers for this purpose, and thus the numbers 3, 5, 8 ; 5, 8, 13 can be placed in columns, since

$$A_{13}=13.5-8.8=+1\,;\quad A_{23}=13.3-8.5=-1\,;\quad A_{33}=8.3-5.5=-1.$$

To complete the determinant, we insert the third column such that the sum of the elements in each row is equal to N. For example, for $N=20$ we have

$$\begin{vmatrix} 3 & 5 & 12 \\ 5 & 8 & 7 \\ 8 & 13 & -1 \end{vmatrix} = 20.$$

The proof is obvious and we have laid down the order of the columns for convenience only.

MAX RUMNEY.

2240. *On the Drawing of Curves of the Type $xf(y)\pm yf(x)=0$.*

When $f(x)$ is transcendental it often happens that the usual processes of analysis, through complexities, give little immediate help towards envisaging the forms of curves $xf(y)\pm yf(x)=0$. If the graph of $f(x)$ can be drawn the desired curve can be readily deduced from it by a straight line process which is almost mechanical, and descriptive properties of it often determined. Analysis may then be found useful for computations at particular points.

Taking first the curve $xf(y)-yf(x)=0$, we notice that the line $x=y$ is part of it. Further, if P and Q are two points on $y=f(x)$, with x-coordinates p and q, and collinear with the origin, O, then the points $(p,\ q)$ and $(q,\ p)$ lie on $xf(y)-yf(x)=0$, and all that is needed to describe the desired curve is an orderly system of pairing typical points P and Q. For the curve $xf(y)+yf(x)=0$ two subsidiary curves $y=f(x)$ and $y=-f(x)$ are generally required with the points P and Q one on one and one on the other.

We give as an illustrative example the curve $x\sin y-y\sin x=0$. Clearly the four lines $x=0$, $y=0$, $x\pm y=0$ are part of the curve, which is symmetrical about them, and it will be sufficient to investigate the part of the curve lying in the first octant. Suppose $y=\sin x$ meets the positive x-axis in the points O (the origin), C_1, C_2, ... in this order and that the tangents from O to the curve touch it at the points with negative y at A_1, A_2 ... and with positive y at B_1, B_2 ... so that C_{n+1} lies between A_n and B_n. Suppose that OA_n meets $y=\sin x$ between O and A_1 in a_n and OB_n meets it in this range in b_n. We will take the points P and Q on the curve, collinear with O, in the order O, Q, P so that for

the first octant we require the point (p, q). As P moves from A_1 to C_2, Q moves from A_1 to C_1 and we obtain the portion of the curve from the point (α_1, α_1) where α_1 is the x coordinate of A_1 to $(2\pi, \pi)$. It cuts $y=x$ at right angles where it is concave when viewed from O but the curvature rapidly changes sign. As P moves from C_2 to C_3, Q moves from C_1 to b_1 and back to C_1 and we obtain a half-wave between $(2\pi, \pi)$ and $(3\pi, \pi)$. Proceeding we obtain a succession of half waves between points $(n\pi, \pi)((n+1)\pi, \pi)$, for P between C_n and C_{n+1}, with end gradients $(-1)^{n-1}/n$ and $(-1)^n/(n+1)$. The points of maximum displacement from the line $y=\pi$, corresponding to P at $B_1, A_2, B_2, \ldots$ are to the left of the " centres " of the waves but tend towards them, with diminishing displacement as the waves proceed. Taking now Q in the range $B_1a'_2$, where $0A_2$ meets $y=\sin x$ in a'_2 between A_1 and B_1, and starting with P at B_1, we obtain a second curve of similar nature tending towards the line $y=2\pi$, and there are in fact an infinite number of these " tendrils ", one starting from each point on $y=x$ whose x-coordinate equals that of an A or a B and tending towards $y=k\pi$ where $k\pi$ is the x-coordinate of the C immediately previous to the particular A or B, the gradient at $(n\pi, k\pi)$, an end of a half wave, being $(-1)^{k+n}k/n$.

The use of squared paper of a generous size and transparent rules and set-squares greatly adds to the rapidity of construction. When the OQP line cuts the subsidiary curve in several points it may be noted that points on different " tendrils " corresponding to various Qs for the same P may be marked in at the same time. The scales for y and x for the final curve and for x for the subsidiary curve(s) should be the same : that for y for the subsidiary curve(s) may be adjusted to convenience. H. GWYNEDD GREEN.

2241. *On the Conditions for a Quadric of Revolution.*

The proofs for the conditions that a quadric should be one of revolution, as usually given, are somewhat indirect and frequently omit the discussion of the sign system after the extraction of square roots. They also invite the query as to why what is effectively the condition that a certain cubic equation should have two roots equal is a double one. The following proof makes direct appeal to the form of the surface, is self-contained, and avoids the points mentioned above.

The necessary and sufficient condition that a quadric should be a surface of revolution is that its intersections with any two planes perpendicular to a fixed line (the axis) lie on a sphere, or, in the usual notation, that l, m, n can be found such that

$$S-(lx+my+nz+p)(lx+my+nz+q)=0$$

is a sphere for all p and q. We must therefore be able to determine l, m, n to satisfy

$$a-l^2=b-m^2=c-n^2\neq 0, \qquad \text{.............................(1)}$$

$$f-mn=g-nl=h-lm=0. \qquad \text{.............................(2)}$$

Hence if none of f, g, h is zero, from (2)

$$l^2=\frac{gh}{f}, \quad m^2=\frac{hf}{g}, \quad n^2=\frac{fg}{h},$$

and, from (1) the required conditions are

$$a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h}\neq 0.$$

The centre of the sphere is given by :

$$2kx=(p+q)l-2u,$$
$$2ky=(p+q)m-2v,$$
$$2kz=(p+q)n-2w \quad \text{where } k=a-gh/f,$$

and the equation of the axis of revolution is

$$\frac{kx+u}{l}=\frac{ky+v}{m}=\frac{kz+w}{n},$$

or
$$f(kx+u)=g(ky+v)=h(kz+w).$$

If say h is zero, then either l or m is zero. If we suppose l is zero, then from (2) as a first condition we must have $g=0$, and (1) gives the second condition $(a-b)(a-c)=f^2\neq 0$, and the equation of the axis in the form (in general)

$$\frac{ax+u}{0}=\frac{ay+v}{m}=\frac{az+w}{n},$$

can be obtained as before.

The examination of the surface by inspection of the section by the inaccessible plane is not without interest. For a surface of revolution we must have a conic (which may degenerate to two distinct straight lines) having double contact with the absolute conic, the pole of the chord being a point on the axis of revolution. If the section degenerates to two coincident straight lines (where the inequalities $\neq 0$ are replaced by $=0$) the surface cannot be one of revolution. If the conic coincides with the absolute conic ($l=m=n=0$) the surface is a sphere. H. GWYNEDD GREEN.

2242. *On the inverse Trigonometric functions.*

The adjustments which have to be made so frequently in calculations involving the inverse functions where their interpretation is allowed as multi-valued suggest that for elementary students at any rate precise single-valued definitions should be adopted, and the following set of rules are put forward as an attempt to standardise the work for such students.

Definitions.

$\sin^{-1} x$ is the angle between $-\pi/2$ and $\pi/2$ whose sine is x: as x increases from -1 to $+1$ through 0, the angle increases from $-\pi/2$ to $\pi/2$ through 0;

$\cos^{-1}x$ is the angle between 0 and π whose cosine is x: as x decreases from $+1$ to -1 through 0 the angle increases from 0 to π through $\pi/2$;

$\tan^{-1} x$ is the angle between $-\pi/2$ and $\pi/2$ whose tangent is x: as x increases from $-\infty$ to $+\infty$ through 0 the angle increases from $-\pi/2$ to $\pi/2$ through 0.

The definitions for the other three ratios follow immediately.

The addition formula for the $\tan^{-1}$ becomes:

If $1-xy$ is positive,

$$\tan^{-1}x+\tan^{-1}y=\tan^{-1}\frac{x+y}{1-xy}.$$

If $1-xy$ is negative,

(i) if x and y are positive,

$$\tan^{-1}x+\tan^{-1}y=\pi-\tan^{-1}\frac{x+y}{xy-1};$$

(ii) if x and y are negative,

$$\tan^{-1}x+\tan^{-1}y=-\pi+\tan^{-1}\frac{-(x+y)}{xy-1}.$$

Wherever practicable it is usually convenient for precision and manipula-

tion to transform an inverse function into its $\tan^{-1}$ form with the angle in the first quadrant.

It should be observed, and given an emphasis that the writer has not seen in text-books, that general identities are, in the main, only true for restricted ranges of the variables, often needing adjustments both in modulus and sign, and that the solution of an equation needs back-check to reject false roots, thus for example proceeding in the ordinary way to solve the equation

$$\tan^{-1}2x + \tan^{-1}x + \tan^{-1}3 = 0,$$

we obtain $x = 1$ or $-1/2$, but $x = 1$, under our definition, is the root of the equation

$$-\pi + \tan^{-1}2x + \tan^{-1}x + \tan^{-1}3 = 0.$$

H. GWYNEDD GREEN.

2243. *An application of the Fourier transform convolution.*

1. The convolution theorem of Laplace transform theory has received wide application for the purpose of the derivation of integral formulæ, involving special functions. It is believed that the corresponding theorem in Fourier transform theory has not been so widely used in this connection and the purpose of this note is to sketch a particular application.

2. By $\mathfrak{F}_u\{f(t)\} = F(u)$ we denote the Fourier transform of $f(t)$, that is,

$$F(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iut} f(t)\, dt.$$

The notation $\qquad \mathfrak{F}_u^{-1}\{F(u)\}$

represents the inverse operation.

The convolution theorem in question is, then,

$$F(u)\, G(u) = \mathfrak{F}_u \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(s)\, g(t-s)\, ds \right\}. \quad \text{.................. (i)}$$

3. From the Sommerfeld integral representations for Hankel functions (Watson, *Bessel Functions*, § 6.21) it appears that

$$\mathfrak{F}_\nu\{e^{ix\,\mathrm{ch}\,t+\lambda t}\} = (\pi i/\sqrt{2\pi})\, e^{\frac{1}{2}\pi i(\lambda+i\nu)} H^{(1)}_{\lambda+i\nu}(x), \quad \text{.................. (ii)}$$

and

$$\mathfrak{F}_\nu\{e^{ix\,\mathrm{ch}\,t-\mu t}\} = (\pi i/\sqrt{2\pi})\, e^{\frac{1}{2}\pi i(\mu-i\nu)} H^{(1)}_{\mu-i\nu}(x), \quad \text{.................. (iii)}$$

where x is real and positive, λ, μ, ν are all real and $|\lambda| < 1$, $|\mu| < 1$.

Hence by (i), if $|\lambda+\mu| < 1$,

$$\mathfrak{F}_\nu^{-1}\{-\tfrac{1}{2}\pi e^{\frac{1}{2}\pi i(\lambda+\mu)} H^{(1)}_{\lambda+i\nu}(x) H^{(1)}_{\mu-i\nu}(x)\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\,[ix\{\mathrm{ch}\, s + \mathrm{ch}\,(t-s)\} + \lambda s - \mu(t-s)]\,ds,$$

$$= (\pi i/\sqrt{2\pi})\, \{\exp \tfrac{1}{2}t(\lambda-\mu) + \tfrac{1}{2}\pi i(\lambda+\mu)\} H^{(1)}_{\lambda+\mu}(2x\,\mathrm{ch}\,\tfrac{1}{2}t). \quad \text{......... (iv)}$$

Formula (iv) is equivalent to the following relations :

$$i \int_{-\infty}^{\infty} H^{(1)}_{\lambda+i\nu}(x) H^{(1)}_{\mu-i\nu}(x) e^{-i\nu t}\, d\nu$$

$$= \int_{-\infty i}^{+\infty i} H^{(1)}_{\lambda-s}(x) H^{(1)}_{\mu+s}(x) e^{st}\, ds$$

$$= 2e^{\frac{1}{2}t(\lambda-\mu)} H^{(1)}_{\lambda+\mu}(2x\,\mathrm{ch}\,\tfrac{1}{2}t), \quad \text{............................ (v)}$$

where $|\mu+\lambda|>1$;

$$H^{(1)}_{\lambda+i\nu}(x)H^{(1)}_{\mu-i\nu}(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\exp\tfrac{1}{2}t(\lambda-\mu+2i\nu)\,.\,H^{(1)}_{\lambda+\mu}(2x\,\mathrm{ch}\,\tfrac{1}{2}t)\,dt, \quad \ldots\text{(vi a)}$$

or

$$H^{(1)}_{\lambda+\nu}(x)H^{(1)}_{\mu-\nu}(x)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\exp\tfrac{1}{2}t(\lambda-\mu+2\nu)H^{(1)}_{\lambda+\mu}(2x\,\mathrm{ch}\,\tfrac{1}{2}t)\,dt, \quad \ldots.\text{(vi b)}$$

where in the last integral λ, μ, ν may be complex, provided $|\,\mathrm{R}(\lambda+\mu)\,|>1$.

4. It is interesting to observe that, by means of the same technique, and, by starting from the same Sommerfeld integral representations for the Hankel functions, formulæ such as (Watson, *Bessel Functions*, § 13·73, (4) and (5))

$$H^{(1)}_{\mu}(z)H^{(2)}_{\nu}(z)=\frac{4}{\pi^2}\int_{-\infty}^{\infty}e^{(\mu+\nu)t}K_{\nu-\mu}(2z\,\mathrm{sh}\,t)\,dt$$

can be derived with very little trouble. B. W. Conolly.

2244. *Approximation to* $\sqrt{x}$.

A method which is related to those described in note 2088 (December, 1949), but which is simple enough to be clear even to the not-so-bright, is as follows:

To find $\sqrt{2}$. The nearest integer to $\sqrt{2}$ is 1, so $\sqrt{2}-1<\frac{1}{2}$.

By repeated squaring we have

$$-2\sqrt{2}+3<\tfrac{1}{4},\ -12\sqrt{2}+17<16,\ -408\sqrt{2}+577<\tfrac{1}{256},\ \ldots$$

From the last relation we may with negligible error write

$$-\sqrt{2}+\tfrac{577}{408}=0, \quad\text{or}\quad \sqrt{2}=\tfrac{577}{408}.$$

Some time ago I asked a fifteen-year-old if he knew you could find a square root by squaring. On receiving the expected reply I demonstrated exactly as above. His face quite lit up when he saw that such a crazy procedure really did the trick, and so easily too. He was also impressed when I pointed out the Pellian equation, solutions of which are given by the numerator and denominator at any stage. At my suggestion he worked out $\sqrt{7}$, starting with $3-\sqrt{7}$, and soon announced with evident delight that it was 127/48 very nearly, and that $x^2-7y^2=1$ had solutions $x=8$, $y=3$ and $x=127$, $y=48$.

He was now receptive to Fermat's two theorems (the one via solutions of $x^2+y^2=z^2$, the other via the decimals of 1/7, 1/17, 1/19), the relation between the first and second halves of the recurring groups in these decimals (just the thing for impressing your cronies who don't know it!), the queerness of $2^{25}+1$, and other matters. The seed was sown, I think.

It is of course just as easy to see how a cube root is found, or for that matter how a cubic is solved. For instance $\sqrt[3]{2}-1<\frac{1}{2}$, so, writing x for the root,

$$x^2-2x+1<\tfrac{1}{4},\ x^4-4x^3+6x^2-4x+1=6x^2-2x+7<\tfrac{1}{16},$$
$$36x^4-24x^3-80x^2+28x+49=-80x^2+100x+1>\tfrac{1}{256},\ \ldots$$

Convergents to the root are found by solving $p_nx^2+q_nx+r=0$. The analogy with the convergents of a continued fraction is brought out by regarding the latter as solutions of $q_nx-p_n=0$.

Given an equation of any degree with at least one real root, we can in this way form an equation whose degree is one less, and one root of which is very close to a root of the given equation. But no one would assert that for equations higher than cubics this is a good method. H. Lindgren.

2245. *A note on magic squares.*

A magic square consists of a number of integers arranged in the form of a square, such that the sum of the numbers in each row, column and the diagonal

is the same. They have been known since very ancient times. However, it was only in the seventeenth century that De la Loubère gave the first general method for constructing magic squares of odd orders. Since then many new methods have been devised. A general method for the construction of magic squares of even order is still lacking.

In recent times magic squares under various restrictions have also been constructed. For example, we have the bordered square—a square that remains magic even after the successive borders are stripped off, or the Nasik square in which the sum of the broken diagonals is also equal to the magic constant.

Here we propose to consider the construction of magic squares in which the sum of the various digits in each row, column and the diagonal is also constant. (If the sum of the digits of the two main diagonals does not coincide with the corresponding sum of the rows or columns—we shall call such squares " semi-magic "). In order to construct such squares it is necessary to know the sum of the digits of m consecutive numbers beginning from, say, α. For the present we shall restrict ourselves to the case $0 \leqslant \alpha \leqslant 9$, $m \leqslant 99$ and $\alpha + m < 100$. Under these restrictions, only magic squares of order less than nine can be constructed. Assuming the last number of the series to be of the form $(10a+b)$, the sum of the digits of m consecutive numbers is

$$S(\alpha,m) = \tfrac{1}{2} \,.\, 10(a-1)a + \tfrac{1}{2} \,.\, 9 \,.\, 10a + a(b+1) + \tfrac{1}{2}b(b+1) - \tfrac{1}{2}\alpha(\alpha-1)$$
$$= \tfrac{1}{2}m^2 - 9am - \tfrac{1}{2}m + \alpha m + 45a^2 + 45a - 9a\alpha.$$

A magic square of the required type is possible only when m is a perfect square and α and a are such that $S(\alpha, m)$ is divisible by $n = \sqrt{m}$.* The different sets of values of (α, a) that satisfy this condition are, for various values of n :

$n=3$ (0, 0), (1, 0), (2, 1), ... (9, 1).

$n=4$ (2, 1), (5, 2), (7, 2), (9, 2).

$n=5$ (0, 2), (5, 2).

$n=6$ (0, 3), (2, 3), (4, 3), (5, 4), ... (9, 4).

$n=7$ (2, 5), (9, 5).

$n=8$ (3, 6), (8, 7).

$n=9$ (0, 8), (1, 8), ... (9, 8).

No general method for constructing such magic squares is known. However, we give below a few simple examples.

11	4	9
6	8	10
7	12	5

(Semi-magic)

2	15	16	5
9	12	11	6
13	8	7	10
14	3	4	17

16	23	0	7	14
22	4	6	13	15
3	5	12	19	21
9	11	18	20	2
10	17	24	1	8

L. S. Kothari

* It must be noted that though this condition is necessary it is not sufficient.

2246. *On note* 2093 : *skew-symmetrical determinants.*

The proof given in the above note, in common with those in textbooks, does not, to my mind, make sufficient use of the symmetrical " appearance " of the determinants involved.

The essential " squareness " of skew-symmetrical determinants (hereinafter styled s.s.d., surely with ample justification) of even order, looks more obvious in the case of small orders than the somewhat lengthy proofs indicate. I would suggest, therefore, that a proof based on the general form of Jacobi's theorem brings out this point rather clearly, as well as providing a somewhat shorter proof, as follows.

Let Δ_n be a s.s.d. of even order, Δ_n' its adjugate, which is, therefore, also a s.s.d. If M_r' is a minor of Δ_n', having r rows and columns, M_r the corresponding minor of Δ_n, then Jacobi's theorem states that

$$M_r' = M_{n-r}\Delta_n^{r-1}, \qquad \text{(i)}$$

where M_{n-r} is the minor (having $n-r$ rows and columns) complementary to M_r.

Writing
$$\Delta_n \equiv \begin{vmatrix} 0 & b_1 & c_1 & d_1 & \cdots\cdots & k_1 \\ -b_1 & 0 & c_2 & d_2 & \cdots\cdots & k_2 \\ -c_1 & -c_2 & 0 & d_3 & \cdots\cdots & k_3 \\ -d_1 & -d_2 & -d_3 & 0 & \cdots\cdots & k_4 \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ & & & \cdot & \ddots & \\ & & & & 0 & k_{n-1} \\ -k_1 & -k_2 & -k_3 & -k_4 \ldots & -k_{n-1} & 0 \end{vmatrix}$$

then
$$\Delta_n' \equiv \begin{vmatrix} 0 & B_1 & C_1 \ldots & K_1 \\ -B_1 & 0 & C_2 \ldots & K_2 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -K_1 & -K_2 & -K_3 \ldots & 0 \end{vmatrix}$$

where $B, C, \ldots, K$ are the cofactors of $b, c, \ldots, k$ in Δ_n.

Then we have from (i), by choosing $r=2$, and taking the first minor of the second order from Δ_n',

$$\begin{vmatrix} 0 & B_1 \\ -B_1 & 0 \end{vmatrix} = \begin{vmatrix} 0 & d_3 & e_3 \ldots & k_3 \\ -d_3 & 0 & e_4 \ldots & k_4 \\ -e_3 & -e_4 & 0 \ldots & k_5 \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -k_3 & -k_4 & -k_5 \ldots & 0 \end{vmatrix} \Delta_n$$

that is,

$$B_1^2 = \Delta_{n-2} \cdot \Delta_n, \qquad \text{(ii)}$$

where Δ_{n-2} is a s.s.d. of even order $n-2$. Hence, if Δ_{n-2} is a perfect square, then so is Δ_n. But Δ_2 is a perfect square, being

$$\begin{vmatrix} 0 & d_3 \\ -d_3 & 0 \end{vmatrix} = d_3^2.$$

Thus by induction the proof extends to all even n. Alternatively, with the aid of (ii), we can express Δ_n in terms of Δ_2, and the result follows.

C. C. PUCKETTE.

2247. *On notes* 2028, 2092 : *the divergence of Σn^{-s} and Pringsheim's theorem.*

If grouping and integration are thought artificial, why not invoke Pringsheim's theorem? If $s \leqslant 1$, then $n \cdot n^{-s}$ does not tend to zero, and the necessary condition for convergence is violated.

More interesting, I think, than this particular case is the whole question of what is and what is not artificial in proofs. Do students feel that any method which is largely indirect is artificial?

To return to Pringsheim's theorem, Hardy in *Pure Mathematics*, p. 350, says that Abel discovered the theorem, and that Pringsheim rediscovered it. Knopp, in his *Theory and application of infinite series*, p. 124, attributes it to L. Olivier (1827). Hardy and Knopp show its use with the harmonic series Σn^{-1}, one as an exercise, the other as a remark on the insufficiency of $n\,u_n \to 0$ as a test for convergence. The proof which I find easiest to follow is that given by Knopp, and by Francis and Littlewood, *Examples in infinite series, with solutions.*

C. C. PUCKETTE

2248. *Notes on conics.* 15. *Oblique eccentricity.*

The definition $|SP| = e|PM|$ lends itself to an immediate extension. If the line through P parallel to some fixed line cuts the directrix in M_θ, the conic is the locus $|SP| = e_\theta\,|PM_\theta|$, where e_θ, the oblique eccentricity, is $e\,|\cos\theta|$, the angle θ being an angle between the fixed line and the focal axis.

For many purposes the use of this extension amounts to nothing more than anticipating a trivial step in a proof. For example, if PQ is a secant cutting the directrix in R, we can appeal to the oblique eccentricity associated with the direction of the line PQ itself for the equality $|PR|/|QR| = |SP|/|SQ|$, instead of drawing perpendiculars PM_P, QM_Q from P, Q to the directrix and using similar triangles ; all we have really done is to draw the perpendiculars beforehand and store the deduction $PR/QR = PM_P/QM_Q$ for reference.

On the other hand, the oblique eccentricity can tempt us on to new ground. Given a line l at slope θ through a point M_θ of the directrix, the intersections of this line with the conic are its intersections with the Apollonian locus $|SQ| = e_\theta\,|M_\theta Q|$, and we know immediately that an arbitrary line can not cut a conic in more than two points. If $e_\theta \neq 1$, there are points H_θ, H_θ' which divide SM_θ internally and externally in the ratio $e_\theta : 1$, and the Apollonian locus is the circle on $H_\theta H_\theta'$ as diameter.

Let O_θ be the midpoint of $H_\theta H_\theta'$, and let SU, $O_\theta V$ be perpendiculars on the given line. Then the condition for the line l to cut the conic in two points is

$$M_\theta V^2 > M_\theta H_\theta \,.\, M_\theta H_\theta',$$

that is,

$$M_\theta V^2 > M_\theta S \,.\, M_\theta O_\theta.$$

But

$$M_\theta V / M_\theta U = M_\theta O_\theta / M_\theta S = 1/(1 - e_\theta^2).$$

Hence the condition is

$$M_\theta U^2 > (1 - e_\theta^2)\, M_\theta S^2,$$

or in a still simpler form

$$SU^2 < e_\theta^2 M_\theta S^2.$$

If ϕ is an angle between SM_θ and l, this condition is $|\sin\phi| > e\,|\cos\theta|$. If O is any point on l, and if OW, OK are the perpendiculars from O on SM_θ and the directrix, then $|\sin\phi|/|\cos\theta| = |OW|/|OK|$ and the condition becomes $|OW| < e\,|OK|$: the line l cuts the conic if the line SM_θ cuts the circle

round O with radius $e \mid OK \mid$. Thus the use of the oblique eccentricity brings the eccentric circle naturally into the picture.

Again, for a family of parallel lines e_θ is constant and therefore $M_\theta V / M_\theta U$ is constant, and since M_θ and U describe fixed lines, so also does V, which is the midpoint of the chord if the line l does cut the circle : midpoints of parallel chords lie on a line, and the intersection of this line and the line through S perpendicular to the chords lies on the directrix.

If the oblique eccentricity associated with an angle γ is unity, the construction of the conic by means of lines in the corresponding direction has the simplicity of the most elementary construction of the parabola, for the Apollonian locus associated with a variable point M_γ of the directrix becomes the perpendicular bisector of SM_γ. This construction is possible only if $e \geqslant 1$. If we are used to the idea that there is a construction for the hyperbola necessarily simpler than any construction for the ellipse, we are the less likely to be scandalised to find the sphere presently discussed in the language and with the algebra first found appropriate for the one-sheeted hyperboloid.

E. H. N.

2249. *Notes on conics.* 16. *Bicircular generation and tangential properties.*

The spectacular application of the bicircular generation of a conic is to the study of a focus-sharing triad, but the element of choice in the bicircular basis can be put to humbler uses. If a conic is traced by the centre of a variable circle which touches two fixed circles, the tangents at the points of contact with the two guides intersect on the tangent to the conic, and since the tangents to the variable circle are equal, the point of intersection is on the radical axis of the fixed circles. In other words, if O is a point on the radical axis, and OP, OQ are the tangents from O to the conic, the points at which the variable circles round P and Q touch the guides are the points of contact of the tangents from O to the guides themselves and are discoverable immediately.

To utilise this construction if O is an arbitrary point of the plane, we have only to recognise the basis of an appropriate bicircular generation. The power of O for the conic in a direction making an angle θ with the focal axis is $\Pi_0/(1-e_\theta^2)$, where Π_0 is the power of S for the eccentric circle Γ of O and e_θ is the oblique eccentricity $e \mid \cos\theta \mid$. Since the power of O for the conic in a particular direction has nothing to do with the foci, and the eccentricity is the same for S' as for S, the power of S' for the eccentric circle Γ' associated at O with S' must be the same as the power of S for the eccentric circle associated at the same point with S. Hence if Δ is the circle round S congruent with Γ and Δ' is the circle round S' congruent with Γ', the powers of O for these two circles are equal : O is on their radical axis.

This result can be verified immediately from the identity

$$SN^2 - S'N^2 = e^2(XN^2 - X'N^2),$$

where N is any point on the focal axis, but our argument dispenses with crude verification.

If with a definite direction of measurement along the axis the circles are given the radii $e \,.\, XN$, $e \,.\, X'N$, then since the algebraic difference $e \,.\, XX'$ is arithmetically equal to the axis AA', the conic generated bicircularly from Δ and Δ' is the given conic.

If O is any point in the plane, circles round the foci, congruent with the corresponding eccentric circles of O, and given opposite cyclic directions or the same cyclic direction according as O is or is not between the directrices, form a basis for bicircular generation of the conic, and the radical axis of these two circles passes through O.

This theorem provides a setting for the somewhat confusing property

associated with the name of Adams, and for the construction usually derived from that property, but it is the construction that now comes first. If OU is a tangent to Δ, then SU passes through the point of contact P of a tangent from O to the conic. The line through S perpendicular to SU, that is, parallel to OU, is the tangent from S to Γ and cuts OP on the directrix; thus P is determined without ambiguity from OU. If it is P that is known, U is identifiable as the projection of O on SP, and to say that Δ has the same radius as Γ is to enunciate Adams' equality, $|SU| = e\,|OK|$.

Merely for drawing the tangents from O we need only one of the circles Δ, Δ', but by introducing both circles we infer very easily the classical bifocal properties involving pairs of tangents. If OP, OQ are the two tangents, and if a circle round P touches the guides in U, U' and a circle round Q touches them in V, V', the four points U, U', V, V' are on the circle round O which is orthogonal to Δ and Δ':

If P, Q are any two points of a conic, the four focal radii SP, $S'P$, SQ, $S'Q$ touch a circle whose centre is the point of intersection of the tangents at P, Q to the conic.

Moreover, since the tangents to the circle round O at U, V intersect in S, the tangents at U', V' in S', the tangents at U, U' in P, and the tangents at V, V' in Q, the lines OS, OS', OP, OQ are perpendicular to the chords UV, $U'V'$, UU', VV', and therefore $(OS, OP) \equiv (UV, UU')$ and $(OS', OQ) \equiv (U'V', VV')$, the notation denoting congruences of crosses. But since U, V, U', V' are concyclic, $(UV, UU') \equiv (V'V, V'U')$. Hence $(OS, OP) \equiv (OQ, OS')$, or in other words

The tangents OP, OQ are isoclinal with the lines OS, OS'.

It is instructive to compare this proof with the usual half-angle argument.

The reflection of S in OP is a point H in $S'P$, and $(OH, OP) \equiv (OP, OS) \equiv (OS', OQ)$. Hence $(OH, OS') \equiv (OP, OQ)$, and since $|S'H| = 2a$, one angle θ between the tangents from O is an angle in a triangle whose sides are $|OS|$, $|OS'|$, and $2a$, whence

$$\pm |OS| \,.\, |OS'| \cos\theta = CO^2 - (2 - e^2)a^2.$$

For a parabola, the part of circles round S' is played by lines parallel to the directrix, and since the only line-circle in a coaxal system is the radical axis itself, it is the ordinate through a point O which combines with the circle Δ to form a basis for generating the curve. The circle round O which cuts Δ orthogonally on the focal radii to the points of contact P, Q cuts the ordinate through O on the diameters through P, Q. The line through O perpendicular to VV' is the line OK perpendicular to the directrix, and OP, OQ are isoclinal with OS, OK. The reflection of S in the tangent at P is the projection M of P on the directrix, $(OM, OK) \equiv (OP, OQ)$, and therefore $|\cos\theta| = |OK|/|OS|$.

E. H. N.

2250. *On the polar equation of a conic.*

The purpose of this note is to exploit the easily derived equation of a conic in polar form. The writer has found this equation a convenient approach to the cartesian equations of the conics for engineering students and a refreshing way of revising elementary conics.

The polar equation is, in the usual notation,

$$r = l/(1 - e\cos\theta).$$

Substituting $x = r\cos\theta$, $y = r\sin\theta$ we have

$$x^2 + y^2 = (l + ex)^2.$$

(i) When $e=1$, $r_\pi = \frac{1}{2}l = a$ (say),

giving $$y^2 = 4a(a+x).$$

(ii) When $e \neq 1$, $r_\pi = l/(1+e)$, $r_0 = l/(1-e)$,

so that $$r_\pi + r_0 = 2l/(1-e^2) = 2a \text{ (say)},$$

giving $$\frac{(x-ae)^2}{a^2} \pm \frac{y^2}{a^2(1\sim e^2)} = 1, \text{ according as } e \lessgtr 1.$$

These equations give immediately the axes of symmetry and the coordinates of the foci. A. BUCKLEY.

2251. *A proof of Hadamard's theorem.*

The theorem may be stated thus: *given a non-singular square matrix A, the square of the absolute value of its determinant does not exceed the product of the scalar squares of its column-vectors, and is only equal to it when the column-vectors are mutually perpendicular.*

(We define the scalar product of two vectors x and y as $\bar{x}'y$, and we call the vectors perpendicular when this vanishes.)

The square in question is the determinant of the matrix $M = \bar{A}'A$, where M is Hermitian, since $\bar{M}' = M$. It is positive-definite; for, if $y = Ax$, we have $\bar{x}'Mx = \bar{y}'y$. Also, the element m_{pq} of M is easily seen to be the scalar product of the pth and qth column-vectors of A. Hence the theorem reduces to: the determinant of a positive-definite Hermitian matrix M does not exceed the product of its diagonal elements, and is only equal to it when the non-diagonal elements vanish.

Consider the matrix, (positive-definite if x and k are suitably chosen),

$$H = \begin{pmatrix} m_{11} & \dots & m_{1n} & x_1 \\ \dots & \dots & \dots & \\ m_{n1} & \dots & m_{nn} & x_n \\ \bar{x}_1 & \dots & \bar{x}_n & k \end{pmatrix}.$$

For the determinant of this matrix, we have

$$|H| = k|M| - \Sigma \bar{x}_p x_q M_{qp} = k|M| - \bar{x}'Nx,$$

where N is the adjoint matrix of M, so that $NM = MN = |M| I$, I being the identity matrix. If we take $x = Mz$, we have

$$\bar{x}'Nx = \bar{z}'MNMz = |M| \bar{z}'Mz,$$

and is therefore positive, and can only vanish if z (and therefore x) vanishes.

Hence $$|H| < kM, \text{ if } x \neq 0; \quad |H| = k|M|, \text{ if } x = 0.$$

Accordingly, if the theorem holds for n, it holds for $n+1$. But it holds for 2; hence the result. M. F. EGAN.

2252. *Convergence of series and integrals.*

It was shown by M. F. Egan (Note 2036) that $\int_1^\infty f(x)dx$ and $\Sigma_1^\infty f(n)$ converge or diverge together provided that $f(x)$ has bounded variation on the whole interval $(1, \infty)$; a simplified proof has been given by Agnew and Boas (*American Mathematical Monthly*, 56, 677–8 (1949)). This generalization of the classical test no longer requires $f(x)$ to have a fixed sign and so suggests a similar theorem involving summability. Such a generalization was in fact given long ago by G. H. Hardy (*Proceedings of the London Mathematical Society* (2) 9, 126–44 (1911)), except that Hardy used the somewhat weaker condition

$\int_1^\infty |f'(x)|\,dx<\infty$ (the difference is unimportant in applications). (Hardy also imposed the condition that $f(x)\to 0$, but this is redundant because the first condition implies that $f(x)$ approaches a limit, and unless this limit is zero, both the series and the integral are trivially divergent.) It seems worth while to call attention to Hardy's work, since it contains a generalization to higher derivatives as well, and this is important in applications. In particular, two of Hardy's results (stated here in terms of bounded variation instead of an absolutely integrable derivative) are as follows. (i) If either $f^{(2k)}(x)$ or $f^{(2k+1)}(x)$ has bounded variation and $f^{(r)}(x)\to 0$, $0\leqslant r\leqslant 2k$, then the series and the integral converge or diverge together. (ii) If $f^{(2k+1)}(x)$ has bounded variation, $f^{(2k+1)}(x)\to 0$, and $f^{(2p-l-r)}(x)x^{-r}\to 0$ for $r\leqslant 2p-l\leqslant 2k-l$, then the series and the integral are simultaneously summable or nonsummable by the Riesz method (R, n, r) (or, in view of the equivalence of the Riesz and Cesàro methods, by the (C, r) method). For $k=0$ (ii) would require that $f'(x)$ has bounded variation, and here Hardy proves that the original condition, that $f(x)$ has bounded variation, actually suffices.

Both Egan and Hardy illustrate their theorems with $f(x)=x^{-b}\exp(ix^a)$, $0<a<1$, $b>0$. This function has bounded variation if and only if $b>a$. Hence the original theorem shows that the series $\Sigma n^{-b}\exp(in^a)$ converges if $b>a$ and $a+b>1$ (since the integral converges if $a+b>1$). However, $f^{(s)}(x)$ has bounded variation if s is large enough, and so it follows from Hardy's result (i) that the series actually does converge whenever the integral does, *i.e.* for $a+b>1$, without the restriction $b>a$. By using (ii), Hardy shows further that both the series and the integral are summable (C, k) if $ka+b>1$. He also considers $f(x)=x^{-a}\exp(ix^b\cos(\log x))$, $0<b<a<1$.

Hardy also proves the corresponding result (with $k=0$) for Riesz summability $(R, \lambda(n), r)$, with a general $\lambda(n)$, and deduces that $\Sigma(\log n)^p n^{1+ia}$ is summable $(R, \log n, r)$ if and only if $r>p$.

Results for more general methods of summability could be written down, but would involve a certain complexity of statement. I shall prove here the simplest result for Abel summability: if $f(x)$ has bounded variation, the series and the integral are simultaneously Abel summable. We have to show that

$$(*)\qquad \left|\int_1^\infty (e^{-\epsilon x}-e^{-\delta x})f(x)\,dx-\sum_{n=1}^\infty (e^{-\epsilon n}-e^{-\delta n})f(n)\right|\to 0$$

as $\epsilon, \delta\to 0$, $\epsilon<\delta$. Now we have

$$\left|\int_1^\infty g(x)\,dx-\sum_1^\infty g(n)\right|\leqslant\int_1^\infty |dg(x)|,$$

and so the left-hand side of (*) does not exceed

$$\int_1^\infty |d[(e^{-\epsilon x}-e^{-\delta x})f(x)]|$$
$$\leqslant\int_1^\infty (e^{-\epsilon x}-e^{-\delta x})\,|df(x)|+\int_1^\infty |\epsilon e^{-\epsilon x}-\delta e^{-\delta x}|\,|f(x)|\,dx.$$

The first integral on the right approaches zero since $\int_1^\infty |df(x)|$ converges and so is Abel summable. The second integral does not exceed

$$(\epsilon+\delta)\int_1^R |f(x)|\,dx+2^{-\epsilon R}\max_{x\geqslant R}|f(x)|,$$

and so can be made arbitrarily small by taking R large and then ϵ and δ small.

R. P. BOAS, JR.

2253. *A note on parallel axis theorems.*

The following vector proof of the parallel axis theorem for moments and products of inertia is usually considered as an application of dyadic theory, but there is no loss of elegance by using elementary ideas.

Consider $\Sigma m(\mathbf{a}\wedge\mathbf{r})(\mathbf{r}\wedge\mathbf{b})$, in the usual notation, where $\mathbf{a}$ and $\mathbf{b}$ are fixed unit vectors and $\mathbf{r}$ is the position vector of the typical particle m referred to some origin O. Then if $\mathbf{r}_0$ be the position vector of the centre of gravity, G, of the system, and $\mathbf{r}_G$ the position vector of m referred to G,

$$\mathbf{r}=\mathbf{r}_0+\mathbf{r}_G$$

and
$$\Sigma m(\mathbf{a}\wedge\mathbf{r})(\mathbf{r}\wedge\mathbf{b})=\Sigma m\{\mathbf{a}\wedge\mathbf{r}_0+\mathbf{r}_G\}\{\mathbf{r}_0+\mathbf{r}_G\wedge\mathbf{b}\}$$
$$=(\Sigma m)(\mathbf{a}\wedge\mathbf{r}_0)(\mathbf{r}_0\wedge\mathbf{b})+\Sigma m(\mathbf{a}\wedge\mathbf{r}_G)(\mathbf{r}_G\wedge\mathbf{b}),$$

all other terms vanishing in virtue of the fact that $\mathbf{r}_G$ is referred to G. This might be looked upon as the general form of the parallel axis theorem for moments and products of inertia, in that (i) if $\mathbf{a}=\mathbf{b}$, we have the parallel axis theorem for moments of inertia (ii) if $\mathbf{ab}=0$, we have the corresponding theorem for products of inertia.

It is interesting to note that the above theorem is connected with the principle of the independence of translation and rotational motion of a rigid body through the theorem that, if ϕ is a function of $\mathbf{r}$ and its first, second, ... time-derivatives, containing linear and quadratic terms only in these variables, and vector and scalar terms may appear, then in the above notation

$$\Sigma m\phi=(\Sigma m)\bar{\phi}+\Sigma m\phi_G.$$

(ϕ is, in fact, quaternionic in form.)

This theorem is easily proved by considering typical terms and evaluating them as above. The vanishing of certain of the coefficients will lead to the particular theorem mentioned. This result is a vector generalisation of a theorem given by Routh in his *Elementary Rigid Dynamics* (7th edition, p. 11, art. 14).

A. BUCKLEY.

2254. *On some familiar invariants.*

In Note 2046 A. A. Krishnaswami Ayyangar showed how " all the fundamental results concerning a pair of straight lines and a central conic can be visualized in one figure ". This figure, for the conic whose equation is $ax^2+2hxy+by^2=1$, contains the circle on OQ as diameter, Q being $(-2h, a-b)$, and also the line $y=a$. A geometrical interpretation was given of the expressions $a+b$, h^2-ab and $4h^2+(a-b)^2$. The present Note adds nothing new, but is written to draw attention, rather more explicitly, to the geometrical analogues of certain of the invariants of the general equation of the second degree in x and y.

For this equation, any translation or rotation of axes within the plane leaves invariant the expressions $a+b$, h^2-ab and, therefore, $4h^2+(a-b)^2$, where a, b, h have their usual significance. The third of these expressions is of course the square on the diameter of the circle of Note 2046. The first two determine the tangent of the angle between the lines $ax^2+2hxy+by^2=0$. Now consider the circles through O equal to that on OQ as diameter. Since the (fixed) angle between the lines has its vertex on the circumference of such a circle, elementary pure geometry at once gives the result that the chords joining the other intersections of circles and line-pair are equal, and, further, are all at the same distance from the centre of their own circles. These then are the geometrical invariants. To obtain expressions for their lengths one may take advantage of their invariance and evaluate them for the most convenient circle. This in

fact is what Mr. Krishnaswami Ayyangar did in his Note, where it appeared that the line $y=a$ was the necessary chord, distant $\frac{1}{2}\mid a+b \mid$ from the centre, and of length $2\sqrt{(h^2-ab)}$. A. G. SILLITTO.

2255. *On Note* 2066 : *the chameleon.*

It is perhaps not quite fair to claim that the curve

$$x^2-2ixy-y^2=1$$

is a parabola since the vanishing of the discriminant of the terms of the second degree, although necessary, is not sufficient; all pairs of parallel lines satisfy this condition, and the above equation represents such a pair, namely, $x-iy=\pm 1$. However, another metamorphosis of this conic is into a pair of equiangular spirals, for, in polars, its equation becomes

$$r(\cos\theta-i\sin\theta)=\pm 1,$$

that is,

$$r=\pm e^{i\theta}$$

a pair of equiangular spirals with constant angle $\phi=\cot^{-1}i$. F. M. GOLDNER.

[Several comments have been sent on the above curve; those members who have not easy access to early numbers of the *Gazette* may be interested to know that this joke is a pendant to one of G. H. Hardy's. In Vol. IV, p. 14, he points out that

$$(x+iy)^2=\lambda(x-iy)$$

is (i) a parabola, (ii) a rectangular hyperbola, (iii) an equiangular spiral.—Ed.]

2256. *The series* $\Sigma n^p/n!$.

Methods are given in textbooks on algebra for the summation of the series $\Sigma u_n x^n/n!$, where u_n is a polynomial of degree p in n, for example, in Barnard and Child, *Higher Algebra*, p. 314. The polynomial u_n is written in the form

$$a_0+a_1n+a_2n(n-1)+\ldots+a_pn(n-1)\ldots(n-p+1),$$

where the coefficients $a_0, a_1, \ldots, a_p$ are independent of n, and by re-arrangement the sum of the series is obtained as

$$(a_0+a_1x+\ldots+a_px^p)e.$$

Taking the particular case in which $x=1$, and $u_n\equiv n^p$, the sum of this series is, since $a_0=0$,

$$(a_1+a_2+\ldots+a_p)e,$$

where the a's can be found by solving in succession the equations

$$\begin{aligned}1&=a_1,\\ 2^p/2!&=a_1+a_2,\\ 3^p/3!&=a_1/2!+a_2+a_3,\\ 4^p/4!&=a_1/3!+a_2/2!+a_3+a_4,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}$$

This leads to

$$a_1=1,$$
$$a_2=2^{p-1}-1,$$

and generally,

$$a_r=\sum_{r=0}^{r-1}\frac{(-)^s(r-s)^p}{s!\,(r-s)!}.$$

Thus
$$\sum_1^p a_r = \left\{1+\frac{2^p}{2!}+\frac{3^p}{3!}+\dots+\frac{p^p}{p!}\right\}$$
$$-\left\{1+\frac{2^p}{2!}+\frac{3^p}{3!}\dots+\frac{(p-1)^p}{(p-1)!}\right\}$$
$$+\frac{1}{2!}\left\{1+\frac{2^p}{2!}+\frac{3^p}{3!}\dots+\frac{(p-2)^p}{(p-2)!}\right\}$$
$$\dots\dots\dots\dots\dots\dots$$
$$+(-)^{p-1}/(p-1)!$$

The following algorithm has been devised for the sum of the a's. We form the table

1						
1	2					
2	3	5				
5	7	10	15			
15	20	27	37	52		
52	67	87	114	151	203	
203	255	322	409	523	674	877
....						

The formation is as follows :

(i) The first element of any horizontal row is the same as the last element of the preceding row ;

(ii) the mth element $(m>1)$ of any row is obtained by the addition of the $(m-1)$th element of that row and the $(m-1)$th element of the preceding row.

Then the required sum is the last element in the pth row, and the sum of the series is this number multiplied by e. For example,

$$\Sigma n^7/n! = 877e.$$

F. WESTWICK.

2257. *On note* 2111.

In Note 2111 Mr. Craig quotes a proof of the converse theorem for the cyclic quadrilateral property. In note 1246 (Vol. XXI, No. 244) I drew attention to it as given in that excellent little book, *Altes und Neues vom Kreis* (W. Lietzmann : Teubner, Leipzic), together with an adaptation for the twin converse.

It may be difficulties over copyright, or perhaps preoccupation with examination syllabuses, but innumerable, almost identical elementary text-books on geometry have since been published with no mention of this excellent proof, and of yet another for the converse of Pythagoras' Theorem in Dr. Lietzmann's other little book *Der Pythagoreische Lehrsatz*.

Mr. Craig, and I hope others, will be interested that these books and others of Teubner's mathematical list are again available. I recently obtained through Max Hueber, Amalienstrasse 79, Munchen 13, a copy newly printed to replace my lost *Flachenland*, which Teubner had the good sense and enterprise to have translated and published for an absurdly small sum, at a time when *Flatland* left little change from a 10/- note.

J. E. BLAMEY.

1682. Six floors of the offices of the West End bookmaker Douglas Stuart were found to-day to have been flooded during the week-end, as the result of a faulty ballcock. The plumbers said the odds against this happening were " once in a hundred years, or 365,126 to one ".—*Evening News*, November 20, 1950. [Per Mr. G. A. Garreau.]

REVIEWS

Cours de Cinématique. Tome III. (Géométrie et Cinématique Cayleyennes.) Par RENÉ GARNIER. Pp. xi, 376. 3000 fr. 1951. (Gauthier-Villars)

In the first two volumes of this work (not reviewed in the *Gazette*) the author treats the classical theory of kinematics in Euclidean space E_3 by the method of " repère mobile ". In this volume he extends the theory to spaces of constant curvature—elliptic space $\mathfrak{E}_3$ and hyperbolic space $\mathfrak{H}_3$. The main tool used is still the method of " repère mobile ", but this is supplemented by the notion of normalised coordinates, which are Weierstrass coordinates in the case of elliptic space $\mathfrak{E}_3$. This notion appears to be fundamental, and allows the geometry to be developed in a form suitable for subsequent application to kinematics.

The volume consists of five chapters and two supplementary notes. The first chapter (Chapter X in the work) gives a very lucid introduction to the geometry of elliptic space $\mathfrak{E}_3$ and hyperbolic space $\mathfrak{H}_3$, defined by the introduction of an absolute quadric into projective space. Classical theorems of elliptic geometry are first established, followed by a corresponding treatment of hyperbolic geometry. Although the close relation between these two geometries is emphasised, they are treated separately, as their differences in the real domain are important for subsequent applications. The conformal representation of these Cayley spaces on to Euclidean space E_3 is carefully explained.

The remainder of the book gives a systematic generalisation to Cayley spaces of the classical theory dealt with in the two earlier volumes. Chapter XI deals with the kinematics of a point, skew curves, the theory of surfaces, uniform helicoidal motion, and the determination of motions associated with a field of variable velocity. Chapter XII describes motions in the hyperbolic plane $\mathfrak{H}_2$, the corresponding theory for $\mathfrak{E}_2$ being included in the treatment of motions on the surface of a Euclidean sphere given in an earlier volume. Ruled surfaces, " axoïdes ", and related topics are described in Chapter XIII. The last chapter of the book deals mainly with the theory of curvature of envelopes of surfaces. The volume closes with two notes—one on areas and volumes, the other on mechanics in a Cayley space. The chapters and notes are followed by exercises which contain supplementary results not included in the main body of the work. A very satisfactory index and list of contents are clearly set out at the end of the book.

Although primarily a study of kinematics in Cayley spaces, this book contains much of interest to pure geometers, particularly the lucid introductory chapter on elliptic and hyperbolic geometry. T. J. WILLMORE.

Dynamic Plane Geometry. By D. SKOLNIK and M. C. HARTLEY. Pp. xii, 290. \$2.56 ; 18s. 1950. (Van Nostrand, New York ; Macmillan)

This is an attractive book. It differs markedly from British textbooks. The range is the syllabus for school certificate, but it contains little of what is now called Stage A geometry, the work being mainly literal. At the start, there is an abundance of short and easy examples on angles and parallels, designed to give facility in argument, but there are few numerical questions in the first 40 pages. There is a diagram of a protractor on page 3, but few examples of its use ; the compass and set-square do not appear till page 76. These facts are not mentioned as criticisms, but as indications of the character of the treatment. The authors state : " We have tried to broaden the methods of demonstrative geometry through the Gestalt approach. We have supplemented the deductive and static pattern of traditional geometry with

inductive and dynamic exploration. By a *dynamic* geometry we simply mean a study of the parts of space and their relations to one another while they are in motion and changing."

Part I (pp. 1-137) is called " the meaning of a proof " and Part II (pp. 140-271) " patterns in thinking ".

Thus much space is given to discussing in simple language the meaning of an axiom, a definition, an undefined term, and a theorem ; for example there are paragraphs on the addition axiom, the subtraction axiom, the multiplication axiom, the division axiom and the substitution axiom, illustrated algebraically and geometrically. Instances are repeatedly selected from topics of no special mathematical significance to show that procedure in geometry is similar to that in other activities ; for example a " scythe " is defined and the reader is asked to define a molecule, a skiff and a syphon. At a later stage in connection with proportion there is a discussion of " argument by analogy ".

This is not to say that geometry gives way to general knowledge, far from it ; a presentation of a geometrical argument is discussed with care, and innumerable simple riders, with a profusion of diagrams, appear on almost every page.

Perhaps the most striking departure from common practice is the treatment of angle properties of a circle. If O the centre and AB an arc of a circle, the angle AOB is called the degree measure of the arc, and this is written

$$\angle O \overset{\circ}{=} \widehat{AB}$$

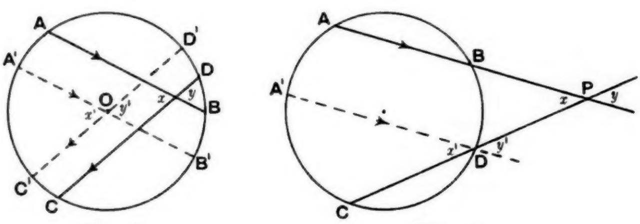


FIG. (i) FIG. (ii)

The fundamental theorem using the notation of Figs. (i), (ii) is :

$$\text{in Fig. (i)},\quad x+y \overset{\circ}{=} \widehat{AC}+\widehat{DB},$$

$$\text{in Fig. (ii)},\quad x+y \overset{\circ}{=} \widehat{AC}-\widehat{BD}.$$

This is proved by drawing the diameters $A'OB'$, $C'OD'$ parallel to the chords AB, CD in Fig. (i) and the chord $A'D$ parallel to AB in Fig. (ii) and using the properties

$$\widehat{AA'}=\widehat{BB'},\qquad \widehat{CC'}=\widehat{DD'},$$

established previously. All the standard theorems on angle properties of a circle, including the alternate segment theorem, are given as *Corollaries* of these results.

The trigonometry of the right-angled triangle based on the cosine, and Pythagoras' theorem, proved trigonometrically, occur towards the end of the book ; the cosine formula for an acute-angled triangle follows and the cosine of an obtuse angle is mentioned in an example. The sine formula for

a triangle is not given, nor is the expression $\frac{1}{2}bc \sin A$ for its area, although there is a somewhat elaborate discussion of areas which includes a carefully drawn 40-sided regular polygon ($2\frac{1}{2}$ inches across), leading up to the conclusion that the circumference of a circle is proportional to the radius.

The varied nature of the examples is illustrated by the following question in a section on the distance of the horizon : " Unlike broadcast waves which follow the curvature of the earth, television waves travel in straight lines. A television transmitter is located on the top of the Empire State Building (1250 feet high). The television waves will strike the horizon before they reach Trenton 60 miles distant. How high an antenna is needed to receive the signal in Trenton? " In this connection, a photograph taken from a V2 rocket 100 miles above New Mexico to show the curvature of the earth is reproduced as the frontispiece to the book.

The lay-out is luxurious, far more lavish than anything that could justifiably be produced at the present time in this country for the home market. This is no doubt the reason why the price is so high that no school could entertain the idea of adopting it as a standard textbook. It should, however, be added to the school library, because it contains much material of value for class discussion. C. V. DURELL.

Analytic Geometry and Calculus. By HAROLD J. GAY. Pp. vii, 524. 42s. 6d. 1950. (McGraw-Hill)

This book covers the material usually included in a first course in both analytic geometry and calculus. It may be divided into three parts, the first of which deals with elementary analytic geometry as far as the conic sections. Graphs and locus problems are dealt with very fully, so that they may be used for illustrative purposes in the second part when introducing the calculus. The exponential, logarithmic and trigonometrical functions are also introduced in this part. The third part is a combination of the two subjects where applications of each to the other are considered, and includes a discussion of three-dimensional analytic geometry, infinite series and elementary differential equations.

The explanations are unusually complete. For example, the idea of a limit is gradually developed before introducing the calculus, and the author has taken great care to point out the underlying difficulties. He has devoted a small but adequate chapter to Indeterminate Forms, in which he considers the meaning of such expressions as $\frac{0}{0}$ and $0\,\infty$, etc. Further, this chapter is placed well on in the book, and hence does not interfere with the student when he is learning the more manipulative side of the calculus.

There are a great number of examples, both of practical and theoretical interest, many of which are worked in the text. P. M. H.

Basic Mathematical Analysis. By H. G. AYRES. Pp. xvi, 584. 42s. 6d. 1950. (McGraw-Hill)

This book is written for students of Junior Colleges, of which there are 500,000 in 650 Colleges in America. Students enter at the age of entry which corresponds to that of Senior Colleges. The contents include Algebra, Trigonometry, Calculus and Coordinate Geometry (which the author calls Analytics). The author aims at unifying these subjects into a single body and, to a large extent, he succeeds. The standard of the Freshman entering Colleges in America is so different from his opposite number here that the English student handling this book for the first time would be puzzled. For instance, of the 518 pages of actual text, more than half is of lower standard than that of the General Certificate of Education, Ordinary Level (Alternative Syllabus). Among the various sets of exercises we have the following

types. (i) Multiply $(+4)$ by (-5). (ii) Simplify $xy/3c \div a/by$. (iii) Simplify $a^4 \times a$. (iv) Multiply $(2x-y)(2x+y)$. (v) Solve the equation $\frac{1}{2}y+y=15$. (vi) Simplify $\dfrac{3x-2}{9}-\dfrac{2x-3}{2}$.

In this book, as in so many American textbooks, the author writes very much to a set pattern. There is an easy simple style which puts the student at his ease, but in which it is clear that the author is writing down to his reader. Results are given freely without proofs. ("You wouldn't understand the proofs.") This attitude is not without danger. It conduces to a careless attitude in the author, which grows worse and which becomes evident in the use of loose statements and to the giving of "proofs" which involve assumptions. Mathematics, at all stages, must use and inculcate honest thinking. If assumptions are made, they should be clearly stated. In the "proof" for the expansions for $\sin x$ and $\cos x$, for example, among the unstated assumptions we have the limits when $n \to \infty$ of $\left(\cos \dfrac{x}{n}\right)^n$ and $\left[\dfrac{\sin x/n}{x/n}\right]^n$ are each equal to 1. The author repeatedly gives phrases such as "The function $x^2+y^2+Dx+Ey+F=0$"; "The function is a parabola". He defines an asymptote as "The straight line which the curve approaches as the point which generates the curve recedes from the origin without end". We are told that "A projectile from a gun generates a parabola". Elsewhere, in a worked example, accelerations and forces are equated freely. The gradient of a line which is perpendicular to a given line is the "negative reciprocal" of the gradient of the line. Too often, proofs are something to be got over as soon as possible, something "you need not bother about unduly". This attitude tends towards compiling a list of results which the student is expected merely to remember. The effect on the student must be, mathematically speaking, quite deplorable, because he misses altogether the spirit of the subject which is inherent in its development.

The treatment, on the whole, is broad. We have a discussion of scales of notation and of numbers. The meaning of the latter is broadened to include fractions, directed numbers, and lastly complex numbers, although the last receives inadequate treatment. Vectors are also brought in early and applied to the composition of velocities. Pages 153–232 give an elementary treatment of Trigonometry up to the solution of the general triangle. Later we have De Moivre's Theorem, the expansions for $\sin x$ and $\cos x$ and an elementary treatment of the hyperbolic functions.

The Calculus occupies only 22 pages. It deals with the derivatives of simple polynomials and with easy maxima and minima. Three-dimensional Coordinate Geometry is developed side by side with two-dimensional work. The plane follows closely on the straight line, and there is a common chapter for the circle and the sphere. A chapter of 45 pages is devoted to the Conic Sections and another of 18 pages to Quadric Surfaces. It will be clear that the treatment is simple throughout. The equations of the various Quadric Surfaces are given, accompanied by sketches, but there is no explanation of how each is obtained. We are given the tests for finding whether the general equation of the second degree represents an ellipse, hyperbola or parabola, but again without proofs. The equation of the parabola is derived from the focus directrix definition. Those of the ellipse and hyperbola are obtained from the loci $SP \pm S'P =$ constant (anything to simplify the treatment), but here the directrix properties are ignored completely.

The linear function and the coordinate geometry of the straight line are taken together. The problem of fitting a straight line to agree with experimental data is discussed. The important method of least squares is given,

which, unfortunately, is not generally discussed in English textbooks. Other fundamental relationships are discussed, including the quadratic, cubic and exponential, but the subject of finding laws to fit experimental data is not pursued further. The solution of equations (*e.g.* cubics) to any degree of accuracy is dealt with by continued expansion of the graph in the neighbourhood of the solution. Determinants are introduced in connection with the solution of linear equations, but the subject is not developed very far. There is a chapter on Polar Coordinates, and a number of graphs dependent on these are discussed. In another chapter the author deals with simple cases of permutations, combinations and probability.

Historical sketches, of which there are quite a number and which are accompanied by photographs of famous mathematicians, are a good feature of the book. The following errors were detected. On page 407 equation (15–38) should read $C' = A \sin^2 \theta + \ldots$. On page 487 the error of 1·1 (in 11·8) is given as 0·9%.

S. I.

Études de philosophie des sciences. Pp. 175. 1950. (Griffon, Neuchatel)

This collection of essays appears in honour of Ferdinand Gonseth's sixtieth birthday. The eighteen contributions have little in common, but their very diversity illustrates the breadth of Gonseth's interests and the wide field of thought over which his influence has spread.

The first essay, by Gaston Bachelard, paints a picture of Gonseth the man and Gonseth the thinker, the keen but courteous protagonist driven by a scrupulous conscience, which could make no compromise with truth, to return again and again to a revision of principles, seeking security for mathematics in the application of a remorseless rigour.

In the second essay Paul Bernays criticizes the familiar thesis that a mathematical entity exists if the supposition of its existence does not lead to a contradiction. While conceding that non-existence is proved by deriving a contradiction from an assumption of existence, Bernays observes that classical proofs of freedom from contradiction consist in an appeal to the existence of a " model " of the system, so that so far from freedom-from-contradiction proving existence, existence is assumed to prove freedom from contradiction. It is true that the Hilbert *Beweistheorie* envisages the possibility of an intrinsic proof of freedom from contradiction, but Gödel has shown that this goal is unattainable.

Jean-Louis Destouches contributes an account of Gonseth's philosophy in dialogue form. Idoine, Eidétos, Théorius, Positivus and Asinus debate six principles of the new rationalism, with a comment from Semantitski on behalf of the Polish logicians.

As a variation on a theme by Gauss, to the effect that much in the theory of numbers is the outcome of a fortunate generalisation from particular instances, Polya gives an amusing account of the application of experimental technique to the discovery of the relationship between an odd number n and the number of representations of $4n$ as a sum of four odd squares.

The last essay is a review by Jean Rossel of a half-century's progress in physics, and the book concludes with a list of Gonseth's publications, comprising 12 books and 52 articles on geometry, relativity theory and the foundations of mathematics.

R. L. G.

Chambers's Shorter Six-Figure Mathematical Tables. Compiled by L. J. Comrie. Pp. xxvi, 387. 13s. 6d. 1950. (W. and R. Chambers)

To bridge the gap, obvious in both scope and price, between their very handy *Four-Figure Mathematical Tables* and their two splendid volumes of *Six-Figure Mathematical Tables*, Messrs. Chambers and Dr. Comrie have pro-

duced the present volume. Many of the tables are, of course, extracted from the two larger volumes ; some have been reduced by increasing the tabular interval, while some have been sacrificed. The user will be chiefly surprised at the amount it has been possible to retain in one, comparatively small, volume.

As in the parent work, there is an adequate explanation of content and mode of use, and a good deal of auxiliary information. The printing and the clear, open page are as praiseworthy as ever. The reviewer has already written emphatically in commendation of the two-volumed work (*Gazette*, XXXIII, May 1949, pp. 150–2) : this commendation stands, and all that need be added here is a brief statement of the tables given in the shorter book :

logarithms, 1000(1)10000 ;
circular functions and their logarithms, mainly at 1′ interval, with special provision for angles near 0° or 90° ; circular functions with radian argument ;
exponential and hyperbolic functions, natural logarithms, inverse circular and hyperbolic functions ;
powers, roots, reciprocals, factors and factorials, 1–1000 ;
primes up to 12919 ;
formulae for numerical differentiation and integration ;
tables to facilitate interpolation.

It should be emphasised that the price quoted in the heading is correct ; you really can get all this for 13s. 6d. T. A. A. B.

Leerboek der Goniometrie en Trigonometrie. By P. WIJDENES. 7th edition. Pp. 263. f. 13.60. 1950. (P. Noordhoff, Groningen)

Boldriehoeksmeting van J. Versluys. 10th edition. Revised by P. WIJDENES. Pp. 297. Paper f. 9.50. Boards f. 11.50. 1950. (P. Noordhoff, Groningen)

Perhaps our impressions of Continental textbooks are biassed in their favour because only the best find their way here for review. However that may be, it seems that Holland, France and Germany rely on a few " classic " authors, whereas in England we suffer annually from a flood of *ad hoc* publications of Mr. X's lessons or Dr. Y's lecture notes. The volumes under review could be rendered into English to do useful service over here with very little addition or alteration to suit our tastes.

Herr Wijdenes' *Plane Trigonometry* has deservedly reached its 7th edition. In the English textbook calendar it corresponds more to Freeman's *Actuarial Mathematics* than to the usual type, for it is written for a mature and well-instructed public. As with Freeman, the opening chapters are probably redundant unless the Dutch boy habitually starts trigonometric technique in the upper forms of the gymnasium : (I am writing here with a Dutch rather than an English accent). It is good to note the exact terminology : goniometry for the calculus of angle-measurement, trigonometry for its application to geometrical problems. The book begins with sexagesimal, centesimal and radian measure ; then the ratios are carefully defined for acute angles, and the definition broadened to fit all angles. One notes with pleasure that attention is drawn to the etymology of cosine. As usual with Continental books, there is no pigeonholing of algebraic and geometric ideas. The book presents trigonometric technique as a fusion of both. There is an excellent chapter on making formulae " logarithmisch ", *i.e.* amenable to computation with log tables, and this idea is again stressed in the discussion of the half-angle formulae. " English papers, please copy." Too often this part of the course

provides one damned formula after another, with no mention of the unifying principle of " makend logarithmisch ". The author has a rather odd habit of first stating a formula and then amplifying it in " longhand ". This seems unnecessary, for no one deficient in knowledge of algebraic notation could hope to go far into this book, while to those with whom algebra is a concise method of expression one would surmise that to translate $a^2 = b^2 + c^2 - 2bc \cos A$ by " het kwadraat van een zijde van een driehoek . . . e.s.w. " is to render it not merely into Dutch but into double-Dutch. I would humbly suggest to Herr Wijdenes visual aids, *i.e.* the complete geometrical diagrams.

There is a comprehensive treatment of graphical topics, limits and series, and cis θ makes a brief appearance, but strangely enough there is no mention of derivatives and integrals, or the series development of the ratios. The addition formulae are proved in unusual style for acute angles, and then extended for obtuse angles. It would be more in keeping with the author's methods to introduce a more general proof.

Throughout the book the style is easy and lucid; diagrams are numerous and well-drawn and, what is unusual in a Continental textbook, there are many exercises, both as models and set for solution. The concluding chapter by Dr. Meerkens on cadastral survey makes a satisfying ending. The main principles of triangulation, grids and stereographic projections are completely set out, and the resection problem faithfully dealt with.

In the Spherical Trigonometry the preface explains that this tenth edition is an enlargement and revision of a school book, and is now adapted for a wider circle of readers. Like the Plane Trigonometry it is well written and well printed on good paper. It is comprehensive, and concludes with two chapters by Dr. Meerkens on spherical correction tables and transformation of coordinate systems. At first one notes with surprise that there is no mention of the haversine formula, but on consulting two English works in the local library, I found that one entirely misquotes it, and is not too clear on the actual definition of a haversine, while the other, defining haversine correctly, is content with the cryptic statement that navigators find it handy.

Both books have a full index and list of contents, but only Spherical Trigonometry has answers to exercises. The latter also has a list of formulae bound with it; a list for the former book is supplied in pamphlet form. At current rates both books are excellent value at prices between £1 and 25/- To sum up, the authors have been fortunate with their publisher, while he is equally fortunate in them. J. E. B.

A Second Course of Mechanics and Properties of Matter. By A. E. E. McKenzie. Pp. viii, 232. 11s. 6d. (Cambridge University Press)

This book is not designed for the mathematical specialist, but that does not mean he would not gain much from a careful reading. It is written from the standpoint of physics, and is intended primarily for science students in sixth forms who are working to University Scholarship standard.

The first three chapters of the book deal with statics, hydrostatics and dynamics; the accounts are brief but sufficiently comprehensive for those who have followed an introductory course in mechanics. Newton's laws of motion are clearly explained, and the emphasis the author places on the fundamental importance of the third law is commendable, but unfortunately in the following chapter on rotation it leads him into an unnecessarily complicated account of " centrifugal " and " centripetal " forces. Is it really good physics to say, when discussing the whirling of a stone at the end of a string, that " In the case of an elastic string the existence of the centrifugal force is revealed by the stretching which occurs "? In similar vein we are told later that " the mass of the earth may be considered as concentrated at its centre

and its attraction by the moon *is balanced by centrifugal force* ". Should not this force be included with the other " occult forces ", the removal of which from physics was, as the author rightly points out elsewhere, one of the great achievements of seventeenth-century science? To exorcise it would, one suspects, make for clarity in twentieth-century teaching of mechanics.

The chapter on simple harmonic motion includes the compound pendulum and its use in the measurement of g. The rest of the book is devoted to properties of matter—gravitation, elasticity, surface tension and viscosity. The chapter on gravitation is outstanding with an excellent summary of the history of the subject ; it would, however, have been worthwhile adding (p. 135) that when Newton first tested his theory of gravitation by the motion of the moon he used a faulty value of the radius of the moon's orbit and consequently his work was put aside some nineteen years until the use of new measurements and further mathematical analysis yielded agreement between theory and observation. A small blemish is that in several places the term " density " of the earth is used instead of mean density.

The chapter on elasticity discusses mainly the stretching and torsion of a wire. The old graph of load against extension of a steel wire might have had a companion graph of actual stress against extension, which is of greater physical significance. Again, the impression is given that there are three, not two, independent types of strain ; if so, why find later the relation between the three elastic moduli? The usual experimental methods of measuring these moduli are explained, and a brief account is given of the deviations from Hooke's law and of the plastic deformation of metals.

The methods of measuring surface tension are well described. The final chapter discusses the capillary tube and falling sphere methods of measuring the viscosity of a liquid, and includes a summary of the recent work of Bowden and his school on friction between solid surfaces.

At the end of each chapter there is a collection of examples, taken mainly from Scholarship and Higher Certificate examination papers. J. Topping.

Romance in Arithmetic. By M. E. Bowman. Pp. 160. 6s. 1950. (University of London Press)

This stimulating and informative book by one of the Honorary Secretaries to the Mathematical Association will prove valuable to pupil and teacher alike. Miss Bowman quotes in her preface : " Mathematics is one of the main lines which the creative spirit of man has followed in its development "— and this statement is ably substantiated by her treatment of the subject. Picturesque ancient measurements such as the sixteenth-century rod defined as " the length of the left feet of sixteen men lined up as they left church on Sunday morning " are contrasted with the more scientific unit, the metre— one ten-millionth of the distance from the Earth's Equator to the North Pole.

The work is well produced with good illustrations. It is clearly divided into three main sections dealing with measures of value, weights and measures, and measurements of time. Each section is followed by a simple chronological list tracing from earliest times up to the twentieth century the history of money, units of measurement and the calendar. One up-to-date reference is to the 1948 agreement for standardising nuts, bolts and screws, and an excellent case has been made for the introduction of decimal coinage into England.

The aim of the author seems to have been to prove that measurement far from being dull and prosaic can be full of romance and interest. To this end she mentions when dealing with tons that the tonnage of a ship was originally the number of tons of Bordeaux wine it could carry.

In spite of its wealth of information, this book can be appreciated by

children as well as by adults. Its early chapters will prove particularly absorbing to collectors of coins, and it will be a great asset in any school library as a source book to the student of history as well as to the budding mathematician. Y. B. G.

Exercises in Elementary Mathematics. Book Two. By K. B. SWAINE. Pp. 236. 6s. 1950. (Harrap, London)

This is the second of a series of four books planned to meet the needs of those working in the alternate syllabus to school certificate stage. Book One was reviewed in the *Gazette*, Vol. XXXIV, p. 311.

The opening ten pages comprise a well-set-out and useful summary of Book One. For a reason not clear there is a repetition of six pages on " Directed Numbers ", the final topic in Book One, before the new work, a steady development of mathematics as a subject, again on very sound lines and through well-chosen questions.

The algebra includes more simple equations, simultaneous equations, factors and graphs, while the geometry treats congruency, symmetry, similarity, hence ratio and proportion, and sines and tangents, quadrilaterals, intercepts, areas and volumes.

In all this, the progress is such that, led by the teacher, no child should feel that mathematics is a " difficult " subject. One would particularly mention the work on similarity, ratio and proportion and the sine and tangent. It was pleasing to see a section dealing with plans, elevations and sections of solids, a topic so rarely treated in the mathematics class, or so it would appear from the attempts made at these questions by too many candidates who take the alternate syllabus or other examination papers in which questions on this work appear.

The book is not meant to be a textbook, but a book of exercises. After more than 75 sets of exercises, some with more than 40 questions (one has 30), the author concludes with 20 comprehensive sets of Revision Exercises, all of the same good quality. None of the questions is either long or involved. An older student who has missed the opportunity of an organised school course could make good progress in mathematical knowledge by working through this book, even away from a class or tutor.

This book, like Book One, is worthy of the attention of teachers in all types of schools, for such is the presentation that there is progressive work which will attract and hold the interest of the weakest pupils. E. J. A.

Mathematics for Modern Schools. IV. By T. H. WARD HILL. Pp. 203. 6s. 1950. (Harrap)

This book appears to conclude the course started some three years ago. Two main new topics are introduced—logarithms and trigonometry. There is a chapter on algebra which is mainly revision, and the geometry proceeds a little further and includes some simple examples of plans and elevations together with isometric projection. There is an additional chapter on geometry, which seems to meander somewhat aimlessly. Three chapters are headed " Arithmetic of Daily Life ", and include topics such as Rates and Taxes, and Insurance—all treated in a simple way—which are dealt with in Arithmetic of Citizenship.

Logarithms are presented by the method of " This is how the machine works. Put number A into slot marked A and number B into slot marked B, press button X and out comes the product ". This is probably the best method for Modern School pupils. This being so, the author should not have wasted much time on half-finished explanations, but should have concentrated more on a thorough demonstration of how to manipulate the machine. He

should also have given tests of the work produced to show that the machine really does what is claimed of it. The printing of the logarithm tables is too small and unnecessarily cramped, considering that there is plenty of space to spare. It compares rather badly with that in other textbooks. The use of antilogarithm tables was not wise, particularly when we consider the weakness of the pupils who will use them.

Trigonometry commences with the tangent. A right-angled triangle is drawn. The pupil is then instructed to draw several perpendiculars, to measure the opposite sides and the adjacent sides, and to calculate the value of the ratio

(opposite side)/(adjacent side)

for each perpendicular. It is doubtful whether this method carries any conviction. Here was a chance to apply the properties of similar triangles which have been discussed on previous occasions: this would have been more effective.

Trigonometrical tables are given to two decimal places. We have

$$\tan 6° = 0{\cdot}11 \text{ (error} = 5\%),$$

$\tan 2° = 0{\cdot}03$ (error $= 16\%$) and $\tan 1° = 0{\cdot}02$ (error $= 25\%$). This is simplicity carried to extremes.

The presentation is breezy and the diagrams are attractive, with the exception of the one on p. 97, which is something of a monstrosity. S. I.

School Mathematics. II. By H. E. Parr. Pp. xiv, 458, xxv 10s.; with answers, 10s. 6d. In two sections, without answers, 5s. 9d. each. 1950. (Bell)

Part I of this series was reviewed in the *Gazette*, No. 308. It is a good idea to make the present volume available in two parts of one year's work in each. If a one-year section of a unified course book really does suffice, it must be more economical than the older way, without being too heavy or too big. Some teachers might like answers included in the sections, however.

Again one must praise the excellent type used, until the tables are reached. Here broad margins mock the small and ill-spaced figures. Tables easier on the eye are available, and it is a pity these unpleasant ones have been chosen.

Formal geometry is introduced early in this volume. Whatever differences of opinion there may be, everyone will respect the author's position with regard to this subject, and admire his careful labelling of figures and setting out of theorems. Assumptions 1 to 9, three for angles, two for parallels, and four for congruence, are set out in a clear and interesting manner. After these the usual proofs are given.

" 3-figure " men will be pleased, and " 4-figure " men will frown, when they reach p. 98. For here it says, with regard to tables, " the 4th decimal place is not absolutely reliable . . . give the final answer correct to *three* significant figures ". A warning is, however, given : " Always retain *four* figures throughout the working, and whenever a result is used in subsequent calculation." All will welcome this. Moreover, several sets of answers are given to 4 figures in spite of instructions to read off to three. Those who do not agree with the book's policy on this much discussed matter will be glad of the concession.

There is plenty of good material for all four branches of the subject, but those who think that there is not much wrong with the old order of doing things will not welcome the positions of some of the chapters in the book. In this respect it may not offend as much as Part I. Unless the series is introduced gradually from the bottom upwards, it will not at first be easy to make a one-year section do for one year's work. But the old order means nothing

to the pupil, and perhaps not much to the young teacher. At this stage criticism of this sort should not be too severe. H. B.

Technical Drawing for Schools. Books I, II. By S. H. Glenister. Pp. 128 each. 5s. each. 1950. (Harrap)

Book I is a very useful introduction to the subject. The progressive steps are well conceived and clearly presented, the figures generally being adjacent to the relevant text.

There are a few minor points for criticism. Examples of freehand lettering and figuring would enhance the value of Fig. 8. The centre of a circle should be defined by a +, even in a rough sketch (Fig. 12), the lines continuing well beyond the circumference to avoid confusion with possible object lines (Fig. 148, which is incomplete).

This book can be recommended with confidence for use in Secondary Schools. The publication of Book III is anticipated with interest.

Book II follows the commendable scheme of Book I, the text and drawings being well presented and well within the scope of young draughtsmen in Secondary Schools.

Remembering the importance of centre lines, it is unfortunate that they have been omitted in the orthographic drawings Figures 28 and 29. Apart from the missing centre line, Fig. 29A is incomplete.

The book is well recommended to follow Book I for use in Secondary Schools. F. T. Davies.

Response of Physical Systems. By J. D. Trimmer. Pp. ix, 268. 40s. 1950. (John Wiley, New York; Chapman and Hall)

An Introduction to Servo-mechanisms. By A. Porter. Pp. vi. 154. 7s. 6d. 1950. Methuen's Monographs. (Methuen)

A physical system of the type considered in these two books is characterised by an input function or " forcing " u and an output function or " response " x. The relation between these two quantities can be represented formally by the equation $x = \Phi u$, where Φ denotes the operation of converting the first into the second. If $\Phi(u_1 + u_2) = \Phi u_1 + \Phi u_2$, the system is called linear.

In the majority of practical applications envisaged the relation between u and x can be represented, at any rate approximately, as a linear differential equation with constant coefficients. Now the mathematician has for a long time been familiar with various formal methods for solving such equations completely. His interest in the problem is therefore limited. The physicist and the engineer, on the other hand, are very much concerned with the technique of obtaining solutions and of displaying and interpreting them, and many books, good and bad, have been written on this aspect of the subject during the last thirty years. To do full justice to this topic a potential author requires not only the precision and clear thinking of the trained mathematician, but also a sympathetic understanding of the practical point of view. He must also possess the ability to bring into relief the essential features of the problem and avoid clouding the issue with relatively unimportant detail.

The special techniques which are employed are usually based on the use of input and output functions of particular types. The most familiar method, particularly appropriate for mechanical and electrical vibration problems, is to assume that u is a sinusoidal function represented by the complex number $U \exp jwt$. The particular integral for x, which is sinusoidal and which has the same frequency, can be obtained directly by replacing the operation of differentiation by multiplication by $j\omega$. Using the algebra of complex numbers, the amplitude ratio (gain) and phase difference between x and u can be

deduced quite easily. Alternatively, the engineer's vector diagram can be used. Various graphical methods are available for showing how these characteristics vary with frequency.

More general periodic functions can be dealt with by analysing u into sinusoidal components by means of a Fourier series, finding the response to each component by the above method and using the linearity property to combine these and give the response x as another Fourier series. This approach can be generalised further to certain non-periodic functions by using Fourier integrals.

The other widely-used method appears in many different, seemingly unrelated forms. Although this point is seldom brought out clearly, it is based on functions which are zero for negative values of t, the simplest of which is Heaviside's unit step function. This leads naturally to the operational calculus of Heaviside, or to its modern form, expressed by means of the Laplace transformation.

This method is particularly suitable for studying communication and other electrical systems based on pulses, and also automatic control systems. It can be linked formally with the first method, using sinusoidal functions, by means of the Fourier transform.

Most of the books that have been written base their development of this subject almost entirely on one or both of these techniques. Indeed, some authors have been known to proclaim with pride that a knowledge of differential equation theory is not required. This outlook is to be deplored for several reasons. Firstly, it is doubtful whether any simplification has been achieved by the time a complete discussion has been given. More than that, it is highly probable that the subject has become enveloped in an air of mystery. Whatever methods may be used for obtaining solutions to specific problems, it does seem desirable that every student of linear systems should be familiar with the fundamental nature of the complementary function and the particular integral, or in the engineer's language, the transient and steady state solutions. These ideas are not difficult to assimilate, and should at any rate form a preliminary to either of the other treatments.

Both the books under review show a reaction to these tendencies. Both authors build their treatments round the general differential equation and both dispense with the Laplace transform. To justify this Prof. Trimmer offers the following proposition (p. 241). " A knowledge of classical methods of solving differential equations, plus a moderate knowledge of the algebra of complex numbers, constitutes a necessary and sufficient mathematical equipment for gaining a good insight into the response of physical systems." This appears in an appendix on the Laplace transform, which includes a very fair assessment of the advantages and disadvantages of this method.

Perhaps this is going too far. To a certain extent all three methods are complementary. The Laplace method comes into its own when the functions involved are of discontinuous form, particularly when steps and impulses associated with switching are involved. The changes which take place in the functions concerned during an operation equivalent to closing or opening a switch are determined by assuming that the differential equation continues to hold during this short interval. The Laplace method takes care of this automatically. In avoiding this method both authors are compelled to resort to treatments of this particular type of problem that can only be described as clumsy.

The greater part of Prof. Trimmer's book is concerned with linear systems. The general pattern is to discuss each type mathematically from a general point of view, using as far as possible a uniform and non-committal system of notation. Examples worked in detail are then given. These are taken

from many different fields of technology, including electrical and hydraulic systems, various types of measuring instruments and nuclear chain reactors. First and second order systems are considered in separate chapters, the responses to certain standard types of forcing function being quoted as particular integrals of the corresponding differential equations. The method suggested for obtaining these is that of undetermined coefficients.

Great care has been taken to indicate the significance of the parameters represented by certain combinations of the coefficients of the differential equations. The author has also gone to considerable trouble to show how the main features of the various responses can be represented graphically.

A chapter is devoted to systems of the third and higher orders. This is followed by a digression on the general features of measuring instruments. The next chapter is concerned with a general discussion of " feed back " systems, a topic developed in detail by Dr. Porter in his book. The last three chapters are devoted to brief surveys of physical systems of more general types. The first of these types is called " parametric forcing ", and is characterised by linear differential equations with non-constant coefficients. The second discusses distributed systems, depending on partial differential equations, while the last touches on the fringe of the vast and as yet largely unexplored field of non-linear systems. This chapter, although sketchy, is in a sense the climax of the book, in as much as it points the way to future developments in the subject. The problems involved here are formidable, for although solutions to particular problems can often be found by numerical methods or by means of some machine such as a differential analyser, the method of superposition associated with linear systems is no longer available, and it is therefore not possible to predict the behaviour of the system for a wide variety of input functions.

Prof. Trimmer proclaims, particularly in the earlier chapters, that one of his objects is to make his reader think about the wider aspects of his subject. This is very laudable, but it does not always make for easy reading. Great care has been taken in the details of exposition, particularly in the choice of symbols. As far as possible all the basic functions of t involved are denoted by q, different functions being distinguished by as many as nine different suffixes. There is a further array of symbols based on Q to denote particular values of the q's. There is much to be said in favour of devising a symbolism which is consistent and precise and which can cover subtle shades of meaning, but the advantages to the reader may be offset if the notation is so complex that he has to keep referring back to recall what it means.

In spite of a few minor points to which objection might be raised, this is a good book, and one which can be recommended to any serious student of the subject.

A servo mechanism is an important special type of physical system. Quoting from Dr. Porter, it can be described briefly as an automatic control system actuated by the difference between the desired response and the actual response, and into which some external source of power is introduced. The term covers not only automatic regulators, whose purpose is to maintain a certain output quantity at a constant value, but also systems designed to follow an input quantity which is varying in some prescribed manner.

Much theoretical and practical development work has been done during the past fifteen years or so, inspired mainly by military requirements. This has been published in numerous reports, which for security reasons have not until quite recently been generally available. In recent years, however, several books have been written on the subject in the U.S.A. The present one is the first to appear in this country, and as such it will be widely received. Moreover, Dr. Porter has been closely associated with the development of

the subject from the beginning, and few, if any, in this country are better qualified to write such a book.

Like other physical systems, servo mechanisms may be linear (within limits) or non-linear. Most of the theoretical work has assumed linearity and is based on the techniques discussed earlier in this review, making use particularly of unit step and velocity and of sinusoidal input functions. Dr. Porter develops his subject on conventional lines, beginning with a number of well-chosen practical examples. He then introduces the idea of a transfer function, which is a linear operator expressed as a function of D (that is, d/dt). Particularly to be commended is the use of the 150-year-old symbol D rather than p, which has a slight difference of meaning in the Heaviside calculus, though it is now almost universally used indiscriminately by engineers. Strictly, p was associated with initially quiescent systems, and is now the Laplace transform variable, whereas the use of D implies greater generality, and is therefore more appropriate for general discussion.

There follows a chapter on different methods of determining the response of the system, with special reference to stability and the nature of transients and to steady state dynamical errors. The necessity to limit the size of this book no doubt has led the author to omit or curtail his mathematical explanations. There, however, is one place (§ 3·5) where an extra paragraph or two might have been a great help to a reader with no previous knowledge of the subject.

The next chapter is devoted to harmonic response diagrams, including vector response loci and those which show amplitude and phase relations separately. Of particular interest in the latter category are the characteristics which show log gain as a function of log frequency. These curves have asymptotes which allow their general shape to be estimated very easily, and stability can be ascertained from these by inspection, with the help of a phase locus. The best-known method of studying stability graphically is, however, that of Nyquist, developed some twenty years ago in connection with feed back amplifiers (an analogous problem). This is based on the position of the point -1 in the complex plane in relation to the vector response locus. Harmonic response loci have several attractive features in practice. First of all they can be plotted without difficulty if the transfer function is given, or alternatively they can be derived experimentally from a system whose parameters may be unknown. Secondly, the stability can be investigated without the laborious arithmetic often necessary in solving the auxiliary equation of the differential equation. Thirdly, it is sometimes easy to see how stability can be improved by a suitable modification of the system, making use of the well-known rule for multiplying complex numbers given in modulus and argument form.

In his fifth chapter Dr. Porter introduces some of the special devices used in servo systems, including the application of integral and derivative of error, output damping and subsidiary feed back loops. He concludes with an example of a system designed in detail to a given specification. The last chapter is devoted to some of the principal sources of non-linearity in servos, including static and Coulomb friction, backlash and amplifier saturation.

Although there are a number of minor errors and other criticisms, Dr. Porter has written a sound and readable little book. He has developed his subject carefully and systematically, and a student with adequate mathematical background will find it an excellent introduction to a branch of engineering which is of increasing importance to the present time.

B. M. Brown.

Infinite Matrices and Sequence Spaces. By Richard G. Cooke. Pp. xiii, 347. 42s. net. 1950. (Macmillan)

The theory of infinite matrices has been developed during this century mainly along lines determined by certain specialised applications. Twenty years ago P. Dienes began a systematic investigation, and with the collaboration of the author of the work at present under review, has made infinite matrices a standard subject of post-graduate courses at Birkbeck College, London. During these twenty years many valuable original contributions to the study of this subject have been made by Dr. Cooke, and his book contains a wealth of hitherto unpublished material resulting from research by him, by P. Dienes, and by others.

The book gives the first comprehensive account of the known algebraic properties of infinite matrices and of their application to summability, to Hilbert vector space, and of their relation to sequence spaces. The algebra of infinite matrices involves the use of limit processes, and thus the interest is always two-fold : both algebraic and analytic. The matrix represents and reveals many operations in analysis, and by its simple algebraic mechanism it readily operates in spaces of infinite dimensions. Thus the study of infinite matrices, as developed in the book, provides an eminently suitable equipment for the study of quantum mechanics. Science students will gain from this book easy access to the technicalities of infinite matrices, while advanced students of mathematics will derive from it many concrete illustrations to abstract algebraic and topological systems.

The author modestly remarks that much of the material may be considered isolated and disconnected. This at the present moment seems inevitable to me, but I consider it a most praiseworthy feature of this work that it collects together under a common principle several topics hitherto regarded as disconnected. It is, however, regrettable that lack of space prevented the author from including even more information on the systems of infinitely many linear equations and an account of basic series.

Chapter 1 gives definitions of elementary operations with infinite matrices $A \equiv (a_{nk})$, $(n, k = 1, 2, \ldots, \infty)$, where the a_{nk} are complex numbers.

$$A + B = (a_{nk} + b_{nk}), \quad \lambda A = (\lambda a_{nk}).$$

The product $AB = C$ is defined by $c_{nk} = \Sigma_i a_{ni} b_{ik}$, provided that all the infinite series converge ; the *existence* of a product AB is one of the central problems. The infinite unit matrix is $I = (\delta_{nk})$, the zero matrix $O = (0)$. Row-(column)-finite matrices have a finite number of non zero elements in each row (column). A set of matrices $\lambda I, A, B, C \ldots$ such that $AB, BC, \ldots$ exist and belong to the set, and $(AB)C = A(BC)$, is called an *associative field* ; for example, row-finite matrices. Otherwise products may exist without being associative. Chapter 2 discusses reciprocals. Solutions X of the equation $AX = I$ are right-reciprocals A^{-1}, those of $XA = I$ left-reciprocals ${}^{-1}A$ of the matrix A. An infinite matrix may have no, one, or infinitely many right- or left-reciprocals, but, by a theorem of Dienes, these possibilities are restricted in an associative field. A norm called *bound* $|A|$ is introduced satisfying, in addition to the usual requirements, $|AB| \leqslant |A| \cdot |B|$ and $|a_{nk}| \leqslant |A|$. If, for all sequences A_m of the field, convergence in elements to a matrix A with $|A_m| \leqslant M$ implies that A is in the field and $|A| \leqslant M$, the bound is called *semi-closed*, and then convergence of $\Sigma |B_m|$ implies that of ΣB_m to a sum in the field. Then, if $|I - A| < 1$, $I + (I - A) + (I - A)^2 + \ldots$ is the unique two-sided reciprocal of A in the field. A theorem of G. Polya gives sufficient conditions on a_{nk} to ensure the existence of infinitely many solutions of the system $\Sigma_k a_{nk} x_k = b_n$, $(n, k = 1, 2, \ldots, \infty)$, without restricting the b_n. This is used to prove the existence of infinitely many reciprocals for some classes of matrices. Chapter

3 deals with linear equations in infinite matrices. If A, B are in a field where A^{-1} exists, then $X = A^{-1} . B$ is a solution of $AX = B$, unique in the field if A^{-1} is unique. The equation $AX - XD = O$, (D is a diagonal matrix), important in quantum theory, is then discussed. If A is a lower semi-matrix ($a_{nn} \neq 0$, $a_{nk} = 0$ for $k > n$) with distinct diagonal elements, X exists with arbitrary diagonal elements. Matrices A are constructed with solutions X and D for which a solution exists. Then the equation $AX - XA = I$ is discussed with constructions of matrices A for which a solution exists, and others for which no solutions exist. Chapters 4–8 deal with general problems in summability of sequences and series. The method employed is always the transformation of a sequence $[z_k]$ by a matrix A, or of a series Σu_k by a matrix G, into a sequence $[z_n']$, so that $z_n' = \Sigma_k a_{nk} z_k$ or $z_n' = \Sigma_k g_{nk} u_k$, provided that the series representing the transformation are convergent. If in addition z_n' tends to a limit z', this is regarded as the generalised limit of the sequence z_k, or the generalised sum of the series Σu_k. The matrix A is called a *K-matrix* if $z_n' \to z'$ whenever $z_k \to z$, and a *T-matrix* if $z' = z$. For series transformations G is called a *β-matrix* or a *γ-matrix* respectively under similar conditions. Necessary and sufficient conditions, to be satisfied by the elements of a matrix, are obtained for the matrix to belong to one of these classes. The generalised limit may exist when the real $[z_k]$ or Σu_k is divergent, and then it may have any value between the greatest and lowest value of limiting points. There is no T-matrix which evaluates every divergent sequence. K-matrices form an associative field with the semi-closed bound $|A| = \sup_n \Sigma_k |a_{nk}|$. Consistency, inclusion, equivalence between two T-matrix summation A, B are considered, and in one case found to be dependent on the commutability of the product AB. Another approach is by " absolute equivalence " introduced by the author. The repeated operation $A[B]$ is compared with the single-product operation by (AB), in general not equivalent. The *core* of a complex sequence $[z_k]$ is defined as the intersection of all the convex covers of the sets R_k consisting of the elements $z_k, z_{k+1}, \dots$. Knopp's theorem that for a non-negative T-matrix the core of $[z_n']$ is contained in the core of $[z_k]$, and various extensions by Agnew and others are given. The problem of inefficiency of a T-matrix for Taylor series outside the circle of convergence is connected with the existence of a reciprocal in a field. T-matrices are constructed efficient for Taylor series at isolated points. The class of T-matrices of Mittag-Leffler type are discussed in greater detail, and conditions on the matrix elements obtained which make the matrix efficient in the " principal " and " partial star-domains ", representing there the analytic continuation. The connection between the methods of series and sequence transformations is treated in general, and in some of the applications. The question of the " right " value for the generalised sum of a divergent sequence or series is investigated for Taylor series, when the matrix is " translative ", and from another angle, when a T-matrix has the " Borel property " of summing " almost all " 0, 1 sequences to the value $\frac{1}{2}$. Chapter 9 is devoted to the study of Hilbert vector space σ_2, and Hilbert matrices. The elements of the space are vectors $x = [x_1, x_2, \dots]$ such that $\Sigma |x_k|^2$ converges. A topology is introduced with the norm $\|x\| = \surd(\Sigma |x_k|^2)$, ; it is shown that σ_2 is separable, is of infinite dimensions and contains an orthonormal base. The bilinear form $x'Ay = \Sigma_{nk} x_n a_{nk} y_k$, with vectors x, y in σ_2 and matrix A is defined, the double sum being required to converge in the Pringsheim sense. Following Dienes, the matrix A is defined to be an *H-matrix* (Hilbert matrix) if $x'Ay$ exists for every pair of vectors x, y in σ_2. This is then shown to be equivalent to the usual definition that $x'Ay$ is bounded for every pair of unit vectors. Using this as the bound of the matrix A, H-matrices form an associative field. The chapter concludes with results on convolution, on

reciprocals of H-matrices, and on continuity in Hilbert space. (The definition on page 268, first two lines, should be stated for every *unit* vector x.) Chapter 10 gives an account of the theory of sequence spaces as developed by G. Koethe and O. Toeplitz, and later extended by H. S. Allen. Various types of convergence and limit are considered, and the ultimate results are expressed in two ways : in the notation of sequence spaces under projective convergence, and in that of infinite matrix transformations. Thus, this chapter links together two distinct branches of research. The book ends with the proof of the Banach-Steinhaus theorem and its application to the proof of earlier results which had been obtained by more elementary methods. There is an excellent bibliography, an index of names referred to in the text, and a complete general index.

The book is pleasant to read, self-contained, and can be understood by any reader of graduate standard in mathematics. It is well-produced and excellently printed. In my opinion, Dr. Cooke's book will become the standard textbook in this subject. I am looking forward to the second volume, which promises to develop the theory of linear operators and the spectrum.

P. VERMES.

Operational Calculus. By B. VAN DER POL and H. BREMNER. Pp. xiii, 415. 55s. 1950. (Cambridge University Press)

One can open most books on mathematical subjects anywhere without danger of misunderstanding when conventional terms are used, but this is certainly not so in books on operational calculus, and the reader is therefore well advised to start at the beginning. The book under review deals with the " Laplace Transform " $f(p)$ of $h(t)$, denoted by $f(p) \doteqdot h(t)$ and defined as follows :

$$f(p) = p\int_{-\infty}^{\infty} e^{-pt}h(t)\,dt \quad (\alpha < \operatorname{Re} p < \beta)\ ;$$

hence we have also

$$h(t) = (2\pi i)^{-1}\int_{c-i\infty}^{c+i\infty} e^{pt}\,\frac{f(p)}{p}\,dp \quad (\alpha < c < \beta).$$

The two-sided integral emerges naturally from a transformation of the Fourier integral and the one-sided, hitherto more usual, form can always be introduced by multiplying the original by the " unit function " $U(t)$, which is zero for negative and unity for positive values of t. Thus the two-sided treatment would seem to be of wider generality, even if one does not agree with the author's claim that this treatment is " more rigorous ". On the other hand, the familiar formulae look now rather clumsy, as, for instance,

$$U(t)e^{-at} \doteqdot p/(p+a),$$

and the new limits of validity, mentioned in the formulae above, must be remembered. The p in front of the integral sign must also be noticed ; with this definition the image of the unit function is unity, which is pleasant, but the fundamental composition rule for convolutions (or faltung, none of these terms being mentioned in the book) looks less elegant. As to the sign of equality with its two dots, the reader need not bother to remember which is where, since p is consistently used as the argument of the image and t as that of the original.

After a welcome historical introduction, Chapters II to VII and XI to XII deal with definitions, convergence and operational rules ; Chapters VIII to X and XIII to XVI with ordinary and partial differential, difference and integral equations. The last two chapters are called Grammar (*i.e.* operational rules, 10 pages) and Dictionary (27 pages). The Index is satisfactory.

The book is of post-graduate standard and caters for the pure mathematician, the physicist and the engineer. It attempts the honest task of talking to the pure mathematician in his rigorous language and introducing him to many new things and to new aspects of what he (perhaps) knew. Abel and Tauber theorems and various remarks on analytical prime number theory are particularly worth mentioning. Ample illustrations by means of practical examples are provided and will benefit all classes of readers.

Very few misprints have been found. On p. 80 the graph should be reflected in one of the axes ; in the second and third formulae on p. 135 the inequality sign always holds ; on p. 300 $\Phi(x)$ in the first formula should be taken from the lower case to conform with what follows, and on pp. 340 ff. there is some confusion between W_n and w_n. A curious error on p. 307 has conferred a predicate of nobility on (Carl Gottfried) Neumann, to whom one form of the solution of integral equations is due (John von Neumann is, of course, another distinguished mathematician, at present in U.S.A.).

The fastidious reader will find himself irritated by some clumsiness (" grow indefinitely ", " technics "), pedantry (" wherefor ") and similar flaws of the translation into English (a sentence at the bottom of p. 14 has been left without a verb). But this does, of course, not detract from the quality of presentation. For this, as well as for its contents and printing, the book deserves to be a *sine qua non* for mathematical libraries. The price, though reasonable under present circumstances, is rather high for private bookshelves. S. V.

The Kernel Function and Conformal Mapping. By S. Bergman. Pp. vii, 161. \$4. Mathematical Surveys, No. 5. (American Mathematical Society)

In this survey a lucid account is given of the properties of complex orthogonal functions which have been developed, largely by the author himself, during the last thirty years. Orthogonality for such functions can be defined in a variety of ways. In the first of these $f(z)$ is a single-valued regular function in a bounded domain B and $J_B(f,\bar{f})<\infty$ where

$$J_B(f\ \bar{g})=\iint_B f\bar{g}\,dx\,dy. \qquad\qquad (1)$$

The class of all such functions forms a Hilbert space with the scalar product of f and g defined to be $J_B(f,\bar{g})$. The functions f and g are then orthogonal if their scalar product vanishes. Other methods involve harmonic functions and the Dirichlet integral, contour integrals in place of the area integral in (1), and integrals which arise from elliptic differential equations, the biharmonic equation, and from problems on functions of two complex variables.

It is possible to define a complete orthonormal system of functions $[\phi_\nu(z)]$ $\nu=1, 2, 3, \ldots$, in each of these Hilbert spaces. The initial steps in the theory are similar to those for real orthogonal functions and lead to the Parseval formula and the Riesz-Fischer theorem. The next step is, however, quite different from the real function theory. It is the introduction of the kernel function

$$K(z,\bar{t})=\Sigma\phi_\nu(z)\overline{\phi_\nu(t)}.$$

This function has the important reproducing property which in the case of functions analytic in B with orthogonality defined by (1) is

$$f(z)=\iint_B K(z,\bar{t})f(t)\,dx\,dy.$$

The chief emphasis of this book is upon conformal transformations, and a chapter is devoted to the mapping of multiply-connected domains upon standard domains. The expressions for the Green and Neumann functions

in terms of the kernel function are given, and a number of interesting relations between them are obtained.

It is difficult to do justice to this book in a short review. The methods developed are of such wide applicability, the principles established are so fertile, and the book itself is written so concisely and with such a profound understanding of the subject, that any review is inadequate. For the practising analyst and for the research student this book should prove to be of the utmost value. H. G. E.

Coefficient Regions for Schlicht Functions. By A. C. SCHAEFFER and D. C. SPENCER. Pp. xi, 311. $6. 1950. American Mathematical Society Colloquium Publications, 35. (American Mathematical Society)

Since the beginning of the century much attention has been given to the problem of finding the domains in which the Taylor coefficients of the function $f(z)$ must lie, when $f(z)$ is regular and schlicht in the unit circle. (Schlicht means that $f(z_1)=f(z_2)$ implies $z_1=z_2$.) This book gives a systematic account of a method derived from the calculus of variations. By considering an explicit formula for a function $f_\epsilon(z)$ which is " near " to $f(z)$ and is also schlicht in the unit circle, the authors obtain a differential equation from which many extremal properties of schlicht functions can be deduced.

Although this particular problem is of interest mainly to the specialist, the methods employed have an importance of their own and are likely to be of use in other fields. The style of the book is clear, and as the material is placed in short chapters it is comparatively easy to read. The problems dealt with are very difficult, analytically, and it is not possible to give an adequate account of them in a short review.

The last chapter, by Dr. A. Grad, applies these methods to the problem of finding the region of values of $f'(z_0)$, where z_0 is fixed, $|z_0|<1$, and $f(z)$ is any schlicht function. H. G. EGGLESTON.

Contributions to Fourier Analysis. By A. ZYGMUND, W. TRANSUE, M. MORSE, A. P. CALDERON, S. BOCHNER. Pp. 188. 20s. 1950. *Annals of Mathematics* Studies, 25. (Princeton University Press; Geoffrey Cumberlege, London)

This volume contains six papers on different topics in Fourier analysis. They deal for the most part with functions of more than one variable. Three of the authors are in the front rank of mathematical analysts in the U.S.A. (the other two being younger collaborators), and their names are a guarantee of the quality of the work. It is not clear why the papers should have been collected in a book instead of appearing separately in journals. There is nothing in this volume for the general mathematical reader, and the appeal is only to the specialist. J. C. BURKILL.

Algèbre et théorie des nombres. Pp. 224. 55s. 1950. Colloques internationaux, 24. (Centre National de la recherche scientifique, Paris; H. K. Lewis, London)

During the last few years the French Centre National de la Recherche Scientifique, in collaboration with the Rockefeller Foundation, has arranged a number of international conferences in various branches of Science. Some of these were devoted to topics in Pure and Applied Mathematics, such as Algebraic Topology, theory and applications of Probability, Harmonic Analysis, etc. The latest volume of the Proceedings contains the lectures which were given at the Conference on Algebra and Number Theory held at Paris in autumn 1949. The universal interest in modern algebra and the rapid development and ramification of this subject (which for a long time was, but

is not any more, unduly neglected in our universities) is well reflected in this volume, which contains among many others contributions by Artin, G. Birkhoff, B. Segre, van der Waerden, Zariski, A. Weil, Dieudonné. H. Davenport writes on the Euclidean algorithm in certain algebraic fields, L. J. Mordell on cubic equations in three variables with an infinity of integer solutions, K. Mahler on algebraic relations between two units of an algebraic field, and K. A. Hirsch on infinite soluble groups with maximal condition.

K. A. H.

Ordinary Non-linear Differential Equations in Engineering and Physical Sciences. By N. W. McLachlan. Pp. vii, 201. 21s. 1950. (Geoffrey Cumberlege, Oxford University Press)

Contributions to the Theory of Non-linear Oscillations. (Annals of Mathematics Studies, No. 20, ed. S. Lefschetz). Pp. ix, 350. 40s. 6d. 1950. (Princeton University Press ; Geoffrey Cumberlege, Oxford University Press)

Non-linear differential equations were first studied systematically, in connection with problems in celestial mechanics, by Poincaré and others towards the end of the nineteenth century. In 1920, van der Pol and Appleton found that they occurred in the theory of radio circuits, and this led to a growing interest in the subject. Much work has been done since then, first by a group of Russian mathematicians led by Kryloff and Bogoliuboff, and more recently in U.S.A. and in this country. Meanwhile the range of technical applications has also been growing steadily.

The subject is therefore of interest both to mathematicians (analysts and topologists in particular) and to engineers or physicists. It is to the latter that Dr. McLachlan's book, the first on this subject to appear in this country, is addressed ; as he indicates in the preface, his aim is to introduce the available analytical methods by examples arising from technical problems, and not to discuss the general theoretical background. At this introductory level, the book should be useful to the " practical man " who wants to know how to tackle specific problems ; but mathematicians will find it valuable for showing them what some of the problems are.

After a brief introduction, Chapter I and II deal with equations which can be solved explicitly, whilst Chapter VIII discusses numerical and graphical methods. The remaining Chapters III–VII deal with approximation methods, mainly for " nearly-linear " equations (or equations which can be reduced to this form) of the type

$$\frac{d^2x}{dt^2}+\omega^2x=kf\left(x,\frac{dx}{dt},t\right),$$

where k is small and f is either independent of t (free oscillations) or periodic in t (forced oscillations). Typical examples are van der Pol's equation

$$\frac{d^2x}{dt^2}+k(x^2-1)\frac{dx}{dt}+\omega^2x=0$$

and the equation

$$\frac{d^2x}{dt^2}+ak\frac{dx}{dt}+x+bx^3=ck\cos wt.$$

The characteristic feature of such equations is that their solutions remain approximately sinusoidal, at least for a limited time, and the departure from sinusoidal form (generally of order k) can be calculated quite simply to a first approximation. Once this approximation is obtained, one can discuss the main problem : the existence of " steady states ", *i.e.* of periodic or almost periodic solutions. The methods have been developed in great generality by Kryloff and Bogoliuboff ; the author presents them gradually by considering

special examples, judiciously chosen so that various typically non-linear phenomena such as self-excited oscillations, hysteresis effects and subharmonic oscillations can be introduced and discussed. A non-linear modification of Mathieu's equation is treated in Chapter VII. There are two appendices, one of which treats the stability of periodic solutions by means of Mathieu's equation, and the book concludes with a full and valuable bibliography.

The presentation is clear and accurate on the whole, but there are serious misprints in §§ 5.170–5.174, which should therefore be read with caution. A criticism which might be made is that too much stress is laid on purely analytical methods, so that much space is expended on different but essentially equivalent versions of the basic approximation method. Some of this space could well have been used in explaining the geometrical method of studying the solution curves in the "phase plane" of $\left(x, \frac{dx}{dt}\right)$. This method, besides being useful in practical problems (it could have been used, *e.g.*, in §§ 5.170–5.174, 5.18, 8.30–8.36), can be used to discuss stability and related questions; it is only treated rather briefly in Chapter VIII.

The Princeton volume contains seven contributions to the mathematical theory. Six of these deal with rather special problems, and are for the expert only; the remaining one, an article by Miss M. L. Cartwright on "Forced Oscillations in Non-linear Systems", is an informal survey of recent research and of unsolved problems. This article will be of great value to the specialist (it contains some hitherto unpublished work by the author and J. E. Littlewood), but it can also be recommended to the non-specialist who wants to find out what the subject is about. The second half of the article concerns "nearly linear" equations, and contains proofs that periodic or almost periodic solutions exist in certain cases; thus it supplies some of the theoretical background for the methods described in Dr. McLachlan's book. This is particularly useful, since the work of Kryloff and Bogoliuboff, who have proved similar results, is rather inaccessible. The first half of the article concerns the equation

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = p(t), \text{ with } p(t) \text{ periodic};$$

a typical example is van der Pol's equation with a forcing term

$$\frac{d^2x}{dt^2} + k(x^2 - 1)\frac{dx}{dt} + x = ak\cos\lambda t,$$

where k need not be small so that the equation can be drastically non-linear. With such an equation is associated a topological transformation of the phase plane, and it is shown by analytical arguments that this transformation has certain properties which allow topological arguments to be applied to draw conclusions about periodic solutions. It is interesting that this "transformation method" (due to N. Levinson), and the methods used for the "nearly linear" equation, can both be traced back to the pioneering work of Poincaré. His idea of combining analysis with topology seems still to be a fruitful one, as is also shown by several other contributions in this volume, and the problems raised in Miss Cartwright's article suggest that further progress may depend on work in both fields. G. E. H. R.

Gelöste und Ungelöste Mathematische Probleme aus Alter und Neuer Zeit. By Heinrich Tietze. Band 1, xx, 256 pp. (5 plates); Band 2, iv, 305 pp. (4 plates). 1949. (Biederstein Verlag, München)

In these two volumes Professor Tietze has gathered together fourteen lectures which he gave "für Laien und für Freunde der Mathematik". The

subjects discussed are: prime numbers and prime pairs, geodesics on surfaces, trisection of an angle, neighbouring regions (with an introduction to the topological classification of surfaces), squaring the circle, three (and more) dimensions, the regular 17-gon, the solution of equations by radicals, the four colour problem, infinity in mathematics, Fermat's last theorem, and the curvature of space. The lectures are profusely illustrated and the exposition is detailed and lucid. There are extensive historical references, including biographical notes on some of the most famous mathematicians. An interesting feature is the set of portraits of the following: Abel, Cantor, Cayley, Dedekind, Descartes, Dirichlet, Euler, Fermat, Galois, Gauss, Hilbert, Klein, Lagrange, Leibniz, Lindemann, Lobatschefski, Möbius, Newton, Emmy Noether, Pascal, Poincaré, Riemann and Tschebyschef.

The present work fills a gap in the literature which expounds mathematics for audiences other than professional mathematicians. Professor Tietze's presentation is of such a high standard that one cannot but regret that the lectures are only available in the German language; and a translation into English seems desirable, in order to bring the subject-matter to a wider audience. The style is charming and, as one reads further and further, one becomes steeped in an atmosphere of peace and wonderment. What is mathematics? In this book one may find a satisfactory answer to this question. The beauty and purity of mathematics is revealed on almost every page, and the absoluteness of mathematics is evident throughout.

My first few hours of reading this book produced an unusual experience. I suppose most mathematicians spend the greater part of their time immersed in research of one form or another. It is an enjoyable experience, but I believe we sometimes miss much by not pausing to contemplate and thereby enjoy the wealth of material which has accumulated in the past. When I came to Professor Tietze's book I realised this more deeply. With this book I was compelled to read about some of the elementary but fundamental facts of mathematics, and of some of the outstanding unsolved problems, which have presented a challenge for generations. And when the reading was finished I found myself in an atmosphere, which recalled the opening sentences of a recent radio talk by Professor Temple,* on "The Nature and Charm of Mathematics". He said: "" Mathematics is a peaceful science. . . . There is no place . . . for passion or prejudice. There is no room for rhetoric, no power in mere persuasion. The theories of mathematics are either true or false, independently of our wishes and opinions." I was indeed filled with this sense of peace and absoluteness. L. S. G.

Einführung in die Höhere Algebra. By GÜNTER PICKERT. (Pp. 298. DM. 12.80; geb. DM. 14.80. 1951. Studia Mathematica/Mathematische Lehrbücher Bd 7., (Göttingen; Vandenhoeck & Ruprecht)

The author aims at introducing the reader to some of the topics of modern research in abstract algebra. The book is primarily intended for students who have already attended a First Year Course, although not much factual knowledge is assumed and all the fundamental concepts are carefully defined.

Emphasis is laid on general ideas and methods. Thus from the outset the author employs the technique of homomorphic mapping to unravel the details of algebraical structure.

About one-third of the book is devoted to a brief survey of the most important types of algebraical structure, including the set of integers, groups, rings, fields and lattices. Since in a book of moderate size it is impossible to give an exhaustive account of all branches of algebra, the author decided to treat in more detail those topics which are necessary for a deeper understanding

* G. Temple: *The Listener*, April 6, 1950.

of the theory of fields. Accordingly, the survey is followed by a discussion of domains of integrity, Euclidean rings, prime elements and prime ideals. A chapter on groups includes decomposition into direct factors and the basis theorem for Abelian groups. The last two chapters, which make up nearly one-half of the book, contain an able exposition of the theory of fields and their valuations. This includes a treatment of inseparable fields, perfect fields, transcendental extensions, the composites of fields and crossed products, with a brief reference to abstract homology theory. Archimedean and non-Archimedean valuations are discussed in the last chapter, which ends with a treatment of valuations of finite algebraic extensions.

The book can be recommended to all students of algebra who have reached a certain degree of mathematical maturity. A valuable feature is the inclusion of numerous examples of various grades of difficulty, some of them dealing with such important matters as proving the fundamental theorem on elementary symmetric functions (a weaker version of which is, in the text, derived from Galois theory). A short bibliography gives suggestions for further reading.

The style is concise, yet clear and the printing very accurate, though a little more generosity with the spacing of formulae would have been more pleasing to the eye. WALTER LEDERMANN.

Advanced Mathematics for Technical Students. II. By H. V. LOWRY and H. A. HAYDEN. Pp. x, 422. 18s. 1950. (Longmans)

A little over five years has elapsed since the publication of Part I, a review of which appeared in the *Gazette*, XXX, p. 177. In this, the second part, the subject-matter covered is that required by students taking mathematics in Part II of an engineering degree, and therefore merits being described as " advanced " much more aptly than that in the first volume.

Once again the authors have succeeded admirably in combining a reasonable standard of pure mathematical rigour (so often lamentably deficient in books of this kind) with a practical outlook so essential to the engineer or technician. Wherever possible the mathematical text has been illustrated by carefully-chosen examples from the realms of thermodynamics, hydrodynamics, elasticity, and so on, so that the " practical " student is never at a loss to see the significance of a particular mathematical technique.

There are, in all, fourteen chapters dealing with the following topics : indeterminate forms ; infinite integrals, including the Gamma function ; moments and products of inertia ; envelopes and evolutes ; differential equations of the first and second order ; simple operational methods and their applications to mechanical vibrations and electric circuits ; difference equations and interpolation ; determinants, matrices and tensors ; simple vector methods in analytical geometry ; spherical trigonometry ; partial differentiation and Jacobians ; line and multiple integration with the theorems of Gauss and Stokes ; Fourier series and harmonic analysis ; analytical dynamics ; virtual work, energy and stability ; and finally elementary partial differential equations and solution in series.

From this it will be seen that a very large number of topics has been discussed in a comparatively short space, but it must be emphasised that the general treatment has in no way suffered as a consequence, and wherever bookwork is, for legitimate reasons, abridged, adequate references have been given for more detailed study.

One or two minor imperfections have come to light during a first reading : in particular, the hypothesis of the first mean-value theorem (p. 28) of the differential calculus has been marred by the omission of single-valuedness, while on p. 267 the definition of a surface integral is logically unsound. Such

instances, however, are almost inevitable in a first edition of a book of this wide and varied scope.

It seems a pity that more emphasis has not been given to the Laplace transform, which is mentioned only briefly (and almost casually) after an investigation of Heaviside's operational technique. The idea of an operator p which is defined initially by its inverse is almost always baffling to anyone but the pure mathematician, so that the more straightforward approach by means of functional transforms, which obviates the necessity of using operators, has everything to recommend it to the "practical" student interested in solving linear differential equations. The dynamical theory is sometimes a little "dated" in presentation, since more vector methods, the groundwork of which has been most carefully prepared, might easily have been employed to considerable advantage.

These few remarks, it will be appreciated, do not amount to much in the way of serious criticism, being for the most part one person's opinion on certain points of detail in a work written by others. It suffices to say, in conclusion, that we have here a thoroughly reliable and recommendable book which is a credit to its authors. It is well-bound, clearly printed, with numerous sets of examples (and answers) and a very adequate index. J. H. PEARCE.

Quantum Mechanics. By A. LANDÉ. Pp. x, 307. 40s. 1951. (Pitman)

There is little doubt that the book under review will serve as an excellent textbook for university students. It is lucid, concise in its presentation, and it does not hesitate to make real contact, where necessary, with related fields. In this respect, so it appears to the reviewer, the present volume differs from many other books devoted to quantum mechanics, and tends to stimulate in its readers a spirit of further enquiry. Four examples of this last characteristic follow. First, in his discussion of diffraction through crystals, the author does not hesitate to introduce the notion of the reciprocal lattice. Secondly, the familiar uncertainty relations between coordinates and momenta, and between energy and time, are discussed on pp. 38–42. But already on p. 46 contact is very rightly made with the uncertainty relations, less familiar but equally important, between the components of the electromagnetic field vectors, in a discussion which is not overburdened by mathematics. (A reference to a paper by Heisenberg on p. 39 might have been balanced here by a reference to the appropriate original papers.) Thirdly, the treatment of quantum statistics includes a reasonably detailed discussion of Bose-Einstein condensation and some properties of liquid helium—a subject not covered even to this extent in some books on statistical mechanics. Lastly, in the discussion of the Dirac wave equation, van der Waerden's concept of spinors is briefly introduced with references to the appropriate literature. It may reasonably be said that this feature, together with an economy of words and space, will prove to be the valuable distinguishing mark between this book and others on the subject.

The list of contents includes, of course, the usual items. Basic experiments are explained and the history of the subject is borne in mind. Standard applications of Schrödinger's wave equation are given, matrix mechanics is discussed fully, and an introductory chapter on atomic and molecular spectra is included. There are chapters on approximation methods, the Pauli principle and quantum statistics. One chapter is devoted to each: the Dirac electron, the quantum theory of radiation and the meson theory. A reasonable selection of references to original papers is given throughout. We have here a most valuable contribution to the teaching of quantum mechanics.

P. T. L.

Introductory Statistics. By M. H. Quenouille. Pp. xii, 248. 30s. 1950. (Butterworth-Springer, London)

The author, of Marischal College, Aberdeen, is Head of the Department of Statistics at the University, and this book is intended as a non-mathematical textbook to serve as an elementary manual, with emphasis on the new methods of interpretation of statistics and their consequences, rather than on long-standing methods of collection and presentation. It is another example of the new type of statistical textbook that has broken away from the traditional sequence of frequency distributions, followed by correlation. Each of the nine chapters consists of a section dealing with some relatively elementary application, with a summary of this at the end of this first part and therefore in the middle of the chapter, to be followed by a second part of the chapter, the expanded application and development of more advanced statistical theory. The effect of this approach has been to make parts of the principles of the simpler work appear rather arbitrary (see, for example, p. 7, about the divisor to obtain the variance from the sum of squares). The examples, though drawn largely from agricultural experiments, include a number based on other applications, for example, Scottish anthropology and meteorology, forestry and nutrition surveys : we were interested to see the percentage fish escape figures once more. The idea of variance is introduced very early (p. 6), and analysis of variance plays a very big part in the treatment. Correlation does not appear until p. 105, and frequency distributions, other than the normal and Poisson, not till p. 162 (and then only to be transformed ; the Pearsonian system is not referred to).

The field covered is wide, and for a particular type of student is probably very useful. It deals with, for example, sampling—the heading of the very first paragraph is Sampling—though the types of sampling—stratified, stratified with variable sampling fraction, systematic, multi-stage—do not get dealt with until the last chapter. It deals also with variance ratios, orthogonality and interaction, curvilinear regression, analysis of covariance, null hypothesis, discriminant functions, probit and serial correlation, the last only very briefly, and there is a useful section on dummy variates. Biased errors of measurement and the rounding tendency are dealt with at the beginning of the chapter on sampling methods. The bibliography of some thirty entries is good and useful, two-thirds of the volumes being British and the rest American, except for one from the United Nations Organisation. It should, however, be pointed out that Yates' *Sampling Methods* is published in this country. There are eleven tables, dealing, inter alia, with the normal curve, Snedecor's F, chi-squared, and random sampling numbers : it is not stated what is the derivation of the last (2000 digits).

The volume is intended to be used as a textbook by students and research workers, but it is not altogether convenient for this purpose. The author and printer in combination have adopted the convention of no point after contractions, with the result that we read, for example, on p. 108, " Depth in % sand " when what is meant is depth in inches, % of sand, or, to take another example, a standard deviation of 6 s (p. 42) instead of a standard deviation of 6 shillings. Similarly, hyphens are omitted and we get such solecisms as lb/acre/day (p. 66) for pounds per acre-day : commas are omitted also (for example, between the two equations $b_t = 0{\cdot}9333$ $b_w = 0{\cdot}6496$ of p. 158). The units are often not stated fully, as in the case of a mean count of 3·52 wireworms per four inch cylinder core (p. 47) or of 7·9 millions bacteria on herrings, scales are not given on graphs (for example, p. 26) and caption headings are lacking in clarity (see, for example, in addition to the examples already quoted, p. 49, where $\frac{1}{3}$ means column 1 ÷ column 3). The vocabulary

is also somewhat at fault in certain places, particularly in the first chapter, and there is an occasional weakness of phrasing. The system of reference is not good. The paragraphs can readily be traced to their proper chapter and thus pages, but the same cannot be said of the numbering of the examples intended as exercises: the illustrative examples are in another series and are not numbered at all. Even when a table has a reference number (for instance, Table 1·2 on p. 4) yet this is not always used (thus on p. 10 the one just referred to is indicated as " the above example ").

Except for the points we have dealt with, the printer has done his work well, though there are a few misprints, and the book is attractively set out. It should prove useful for those who wish to handle in particular analysis of variance for the purpose of interpreting the statistics that they may have in their own special research work. F. SANDON.

Introduction to the Theory of Probability and Statistics. By N. ARLEY and K. R. BUCH. Pp. xi, 236. 32s. 1950. (John Wiley, New York; Chapman and Hall)

The authors of this " elementary study of the subject " are two Danes, one a mathematician (Buch) and one a physicist (Arley), and the book is in the Wiley Applied Mathematics Series, edited by I. S. Sokolnikoff, of California. It is apparently an augmented and revised version in English of the book originally published in Danish in 1946 and already by 1949 in its third edition. The book shows little signs of being a translation, though I do not recognise as King's English the word *mnemotechnically* (p. 17) or the expression " born linear " (p. 201), whilst the American form appears in " chapters 1 through 3 ". The book is printed in U.S.A.; Arley was for a year a Visiting Professor at Princeton University.

The book can be well recommended. The opening chapters deal carefully with the foundations of the theory of probability. A distinction is drawn between the statistical description of a model to represent actual phenomena and reality itself, the book taking the line that laws of nature are valid only for the idealised phenomena. The probability of an event is then regarded as a physical constant related to our experience of the event in a series of observations, just as the weight of an iron bar is determined by a specific rod even though repeated measurements of this weight give more or less different values. From this it follows that if an event has a probability of 1 it is *practically certain*, that is, its relative frequency must be expected to be very close to 1 if the number of observations is very large. From this definition and associated axioms the first three chapters lead up to Bayes' theorem.

Chapters 4 to 8 deal with somewhat the same topics as David's *Probability Theory* (see review, *Math. Gazette*, XXXIV, p. 155): they include therefore Discriminant Functions, Probability Density and Probability Mass, Moment Generating Functions, Characteristic Functions, and other aspects of the mathematical theory, basing the treatment largely on Cramer: it includes the idea of " converging in probability to x ". Although the preface says that a knowledge of calculus is required here, yet this must not be taken in the school textbook sense, for matrix theory and Stieltjes integrals both play their part, and Boltzmann's constant and the quantum theory both get reference.

Chapter 9 then proceeds to tie up the mathematics of the model with nature itself. It points out that we can never ask whether or not the theoretical results agree with the empirical data, but only whether they agree sufficiently well. In Chapter 10 therefore the writers go on to the application of the theory of probability to statistics and then to the question of, for example, finding estimates of the parameters of the mathematical functions.

Kapteyn's class of distributions from the normal is freely used and Fisher's Method of Maximum Likelihood for the practical question of obtaining consistent estimates.

Chapter 11 (Theory of Errors) leads to the idea of tolerance limits and to confidence or fiducial limits (no distinction is drawn between the last two). In Chapter 12 (Theory of Adjustment) matrix theory and notation is used because of its conciseness, and Appendix 2 gives a three-page note on the notation and theory. This chapter, the last, leads to regression analysis.

There are two appendices (the two-page appendix 1 deals with the gamma function) and five tables. These consist of two of the Normal Distribution, one of Student's Distribution, one of Arley's r-distribution (allied to t) and one of w^2 (equivalent to Snedecor's F and thus to Fisher's z). The book closes with ten pages of 90 problems (I think that Qn. 11 is faulty unless the percentages work in integral steps of 1%) and three pages of bibliography: the General Theory references are to workers such as Cramer and Kolmogoroff, the English writers appearing among those working on practical applications. There is a four-page index.

It is a fascinating and cogent book and can be thoroughly recommended: one is not surprised that the original went through three editions, although written and published in a language commanding a small market.

F. Sandon.

Some Theory of Sampling. By William Edwards Deming. Pp. xvii, 602. 72s. 1950. (John Wiley, New York; Chapman and Hall)

The content of the subject of Statistics is by now so extensive that the tendency is to write books specialising in the different topics of applied interest. The author of this text is adviser in sampling to the Bureau of the Budget in Washington, and is a recognised authority on the use of statistical sampling methods. We note, therefore, with interest that there are chapter headings such as The Planning of Surveys, The Various Errors of a Survey, Multistage Sampling, Ratio-Estimates and Choice of Sampling Unit, Allocation in Stratified Sampling, Distinction between Enumerative and Analytic Studies, Control of the Risks in Acceptance Sampling, Inventories by Sampling and A Population Sample for Greece, all of which cover a wide range and convey much authoritative information. But these are only 8 chapters out of a total of 17. Some of the others, although more theoretical, are of related interest, such as Some Variances in Random Sampling and Estimation of the Precision of a Sample. The others quite definitely stray into the field of Mathematical Statistics proper. While these are likely to be of more interest to the mathematician, the question arises as to how necessary much of this material is to the subject dealt with, bearing in mind especially that textbooks on mathematical statistics exist which deal with all the points studied. It is clear that the author has made this a very personal book. He evidently makes a hobby of the mathematics of statistics, while not himself a professional mathematician. This is shown most clearly towards the end, where many detailed presentations are given of advanced mathematical proofs, acknowledgment being handsomely given to others who have in many cases provided those proofs. The mathematician will, therefore, find a mine of information in the book, although in some cases the methods used are not too rigorous, and a sense of proportion is sometimes lacking in that the same essential mathematical proof is repeated, with variations, in more than one place, just because the test that is being discussed is a different one.

In the upshot the book is overlong, and decidedly dear even by American standards, so that the number of mathematicians who will wish to invest in a private copy is bound to be limited. This criticism may even be made more

severe, in the sense that, rightly or wrongly, the non-mathematical reader may imagine that he must know all that is in the book to become an authority on sampling, but instead will flounder because of lack of mathematical equipment. He may also be deterred by the title, which will suggest to some that even this is not the whole story.

The book has been carefully compiled, and the errors which have been detected are mostly slips which will doubtless be put right in a subsequent edition. There are numerous scholarly footnotes, containing many references for further reading. The only table given is one of Fisher's z, which, curiously enough, is treated rather scrappily. For other tests the author prefers to rely on graphs and nomograms. A large number of exercises is provided for the reader.

J. WISHART.

Statistics : an Intermediate Text-Book. Vol. II. By N. L. JOHNSON and H. TETLEY. Pp. xi, 318. 20s. 1950. (Cambridge: Published for the Institute of Actuaries and the Faculty of Actuaries at the University Press)

Vol. I of this textbook was published in 1949 and the issue of Vol. II was awaited with some interest. While the first volume was a satisfactory production, suitable for the purpose for which it was intended, it was, of course, impossible to give an opinion on the work as a whole until the second volume was available. Vol. II confirms the good impression made by the earlier volume, and there is little doubt but that the authors have written a textbook worthy of the best traditions of the actuarial profession.

To the reader who has not specialised in the subject, the mathematics involved in even an intermediate treatise on statistics is apt to be rather frightening. In a book of this nature therefore there must be some compromise between the rigid mathematical aspect and the formal statement of results. The authors have attempted such a compromise and would appear to be most successful when they make no effort to explain the full mathematical theory. It is probably inevitable that the calculus of distribution functions should be discussed before the applications of the results obtained, and Chapter 11, with which the book commences, is likely to appear difficult to the student. The difficulty is not lessened by a rather indirect approach to the subject, although little adverse criticism can be directed to the chapter as a whole. On a minor point, it is unusual to derive the t-distribution from that of χ^2; the methods described by Weatherburn and others seem preferable. The mathematically-minded reader will find Chapter 12 interesting and worthy of close study; but the average student will need to take much for granted.

The multinomial distribution and its application are adequately dealt with in the following chapter, and it is satisfactory to note that emphasis is placed on the fact that the χ's most frequently used in practice are no more than approximations to theory—a point often forgotten by those new to the subject.

The reader who has already some knowledge of the detailed methods of graduation of actuarial data will possibly turn to Chapter 17 in order to see how far his ideas of the principles underlying graduation are in accord with those of the authors. While here and there criticism may reasonably be directed to methods or statements, the general discussion is thoroughly satisfactory, and the chapter is, in fact, as good an explanation of the subject as has appeared for some considerable time. There will, of course, always be differences of opinion on the best approach to various parts of this subject. For instance, it is a little surprising that, considering the frequency with which 2×2 contingency tables occur in practice, no reference is made to the method for obtaining χ^2 given in Fisher's *Statistical Methods for Research Workers* and

elsewhere. In Table 17.9 χ^2 may be taken as

$$\frac{3437(1531 \times 374 - 1133 \times 399)^2}{2664 \times 773 \times 1507 \times 1930},$$

there being no need to find the exact value. Again, the necessity for erring " on the safe side " implied in paragraph 17.11 has not been expressed as happily as it might have been.

The other chapters call for little comment ; they are clearly written and well set out. The student—especially one who is reading for the actuarial examinations—will benefit greatly from a careful study of the book.

H. FREEMAN.

Recherches Théoriques Modernes sur le Calcul des Probabilités. I. Généralités sur les Probabilités. Éléments aléatoires. By MAURICE FRÉCHET. 2nd edition. Pp. 355, xvi. 1950. (Gauthier-Villars, Paris)

The first edition of Fréchet's well-known book appeared in 1937 ; this second edition, long delayed by difficulties of publication, makes a welcome appearance. Readers who know the first edition will find that no major changes have been made, though the author has used the enforced delay to good effect by thoroughly revising and modernising the text.

This volume is but one of the nineteen which together constitute the comprehensive *Traité du Calcul des Probabilités et de ses Applications* originally planned by Émile Borel, in which is enshrined the whole of the work of the modern French school on probability theory and its applications. The book under review is one of the most important parts of this immense and impressive treatise, because it presents in a readable form the modern theory of probability, in the foundation of which the author has played a leading part.

In two short introductory chapters the axiomatic basis of the theory is established by analogy with measure theory. There follows in Chapter III a discussion of the expectation (defined in terms of the Stieltjes integral), the moments, and other typical values of a random variate ; repeated Bernouilli trials ; generating and characteristic functions. Chapter IV deals with Bienaymé's (or Tchebychev's) inequality and its generalisations. The next chapter, which constitutes nearly half the book, very thoroughly examines the convergence of series of random numbers, and provides the most exhaustive treatment of this subject yet available. The last chapter, a new addition, is a summary of recent work on random elements and vectors in Hilbert and Banach-Wiener spaces, and on stochastic convergence. The appendices include a summary on monotone functions, a note by Paul Lévy on the " distance " of two random variates, and a useful bibliography which has been brought up to date.

M. Fréchet writes with precision, clarity, and authority. He introduces each new topic with illuminating discussions of the fundamental ideas which lie behind them, and leads the reader step by step with patience and understanding. We must be grateful that this book is again readily available, since there is as yet no book of similar scope and status written in English.

B. C. BROOKES.

Leçons de Statistique Mathématique. By MAURICE FRÉCHET. II. Recueil d'Exercices de Calcul des Probabilités. Pp. 52. 1949. IV. Les Ensembles Statistiques Renouvelés et le Remplacement Industriel. Pp. 168. 1949. (Tournier et Constans, Paris)

These duplicated typescript notebooks are published to assist students attending M. Fréchet's courses in statistics given at the Sorbonne. The complete set will consist of five notebooks. The first was published in 1946 under

the title of *Exposé préliminaire du Calcul des Probabilités*. The second notebook, which is the first of the two under review, is a collection of exercises intended to illustrate the preliminary course. It contains 53 examination questions, divided into 9 sections, covering a wide range of topics from the drawing of balls from urns to random walks and problems of convergence in stochastic processes. In most cases the exercises are elaborate, with several carefully enunciated steps. No solutions are available yet, but it is stated in the preface that they will be published as the third notebook of the series " dans un ou deux ans ". The second notebook is only of limited interest unless it can be used in conjunction with a course similar to that given in the first notebook, though examiners in search of ideas may be rewarded by studying it.

The subject of the fourth notebook is known in English as " self-renewing aggregates ". This term is perhaps suitable when the theory is applied to problems of population growth, but as applied to the replacement of industrial machines the term " maintained aggregates " might be more appropriate.

The notebook consists of three variations on a theme of Parodi. Having assumed that the life of a given item of industrial equipment was finite and obeyed a discrete probability distribution, Parodi calculated the annual renewal rate required to maintain the items at their original number. Fréchet generalises this problem, and expounds three methods by which this and similar problems can be solved. The generalised problem has applications to demography and actuarial statistics.

The author applies in turn (*a*) elementary algebraic methods, (*b*) the method of characteristic equations and generating functions, (*c*) the method of calculating by intervals. The advantages and disadvantages of the methods are discussed and compared by reference to numerical examples. As the author is able to combine lucidity with close attention to detail, the notebook is useful both as an exposition of an interesting practical problem and as a demonstration of statistical techniques. The main hindrance to comfortable reading is the uncertainty of the typescript in dealing with the many subscripts and superscripts. B. C. Brookes.

Foundations of the Theory of Probability. By A. N. Kolmogorov. Pp. 71, viii. $2.50. 1950. (Chelsea Publishing Co., New York)

The difficult task of making the theory of probability mathematically respectable has attracted mathematicians mainly of French, German, and Russian origin ; in keeping with tradition, British mathematicians, in so far as they have been interested at all, seem to have preferred to develop the applications of the theory. Kolmogorov is known as one of the select band of Continental mathematicians who, in the last forty years, have been striving to strengthen the foundations of the theory of probability by putting it on an axiomatic basis and by linking it with modern mathematical concepts.

This little book is not a textbook, nor is it a complete logical development of the theory. It is a translation of an important monograph which appeared in the *Ergebnisse der Mathematik* in 1933. It is a summary of Kolmogorov's contributions towards the establishment of a more satisfactory theory, and is addressed to specialists.

The author shows how the theory of probability can be closely linked to pure mathematics through the generalised concept of the integral and the theories of measure introduced by Lebesgue. The analogy between the measure of a set and the probability of an event, and that between the integral of a function and the expectation of a random variate was established by Fréchet. Kolmogorov summarises this method of approach and tackles some further problems, *e.g.* the differentiation and integration of expectation with

respect to a parameter, and the establishing of a theory of conditional probabilities. The last section of the monograph consists of results obtained by the author in collaboration with Khinchine on the Weak and the Strong Laws of Large Numbers.

Those who are already familiar with the subject of this monograph will welcome this careful translation which makes more readily accessible a paper to which reference is often made. Readers for whom the subject is new will be interested to know what developments have followed from Kolmogorov's work. A bibliography has been provided, but unfortunately (except for mention of the English translation (1939) of von Mises' *Probability, Statistics, and Truth,* (published in German in 1928) no attempt has been made to add to Kolmogorov's original references, the latest of which is dated 1933. Nevertheless, this English translation of a paper of historic importance is very welcome, as it may help to draw the attention of British mathematicians to a field they have too long neglected. B. C. BROOKES.

Calculus of Finite Differences. By C. JORDAN. Second edition. Pp. xxii, 654. $5.50. 1950. (Chelsea Publishing Co., New York)

This book, first published in 1939, is based on lectures given in Budapest. The first six chapters, comprising more than half the book, deal with symbolic operator methods applied especially to the polynomials and numbers of Stirling, Bernoulli, Euler and Boole. Numerous historical notes are interesting, and seem reliable, and this first section of the book is probably the most valuable. The account of interpolation and table-making in Chapter VII is unrealistic, and the author's statement in his preface that he has written " especially for practical use, with the object of shortening and facilitating the labours of the computer " can hardly carry much weight : in the present state of knowledge of numerical methods, this book is *not* a manual of computing. The sections on numerical integration are curiously placed in the chapter on resolution of equations, neither topic being treated very fully. The volume ends with two short chapters on difference equations, including illustrations from probability theory. It is not stated how the second edition differs from the first : presumably just in the change of publisher.

C. W. JONES.

Principles of Mathematical Logic. By D. HILBERT and W. ACKERMANN. Pp. xii, 172. $3.50. 1950. (Chelsea Publishing Co., New York)

This translation of Hilbert and Ackermann's *Grundzüge der Theoretischen Logik* has been made from the second German edition (1938), by L. M. Hammond, G. G. Leckie and F. Steinhardt, and edited, with explanatory notes, by R. E. Luce, the Managing Editor of the *Journal of Symbolic Logic*. The book has been beautifully printed and produced, and all concerned with its production have rendered a great service to English-speaking students and teachers of mathematical logic.

The translation is noteworthy both for its fidelity and vitality. It is not in any sense a literary translation, for there are words, phrases and constructions which have kept sufficiently close to the original German to offend a critical ear, but it is clear, consistent and unambiguous, and shows every sign of careful thought and meticulous care in its preparation.

There are three points in which the translation has departed from the text of the second German edition. In the statement of the rule of substitution for predicate variables a short qualifying clause (attribured to Professor Church) has been added. Secondly, to meet a criticism by Professor W. V. Quine, a brief paragraph has been inserted in the section on the Skolem normal form, pointing out (for use in a subsequent completeness proof) that

the proof given of the equivalence of any formula and its expression in normal form may be adapted to show that the expression in normal form of a universally valid formula is itself universally valid. The third, and most substantial, departure from the text is in the proof of Gödel's completeness theorem, where a modification in the argument, due to Professor Church, has been introduced.

The emphasis placed on these corrections in the editor's preface to the translation appears to be absurdly exaggerated. If we compare the second German edition with the third (1949) we find that the third edition contains the correct version of the rule of substitution for predicate variables, and also a corrected proof of Gödel's completeness theorem (different from that in the translation). It is true that the third edition does not meet Professor Quine's point about the role of the Skolem normal form in Gödel's completeness theorem, but it is clear from a passage in the second edition itself (p. 94, line 16) that the authors were quite well aware that Skolem's theorem on normal forms could equally well be formulated with respect to satisfiability or universality. R. L. GOODSTEIN.

Foundations of Analysis. By E. LANDAU. Translated by F. STEINHARDT. Pp. xi, 134. $2.75. 1951. (Chelsea Co., New York)

Recently the Chelsea Company issued a reprint of Landau's *Grundlagen der analysis*, and they have now followed this by a translation. It may be wondered whether a translation is really necessary, for the original was an example of the later Landau style at its most terse : Landau himself described it as " unbarmherzigen Telegrammstil ", and it is hard to believe that any student has been frightened off by the need to look up about 50 words in a dictionary. The book itself is, however, so instructive that nothing but good can come from giving it as wide a currency as possible, whether in the original or in translation.

Landau's excuse for this book was that in the whole literature of mathematics there is no textbook which confines itself solely to the task of discussing the operations with numbers on which analysis depends, from the integers, based on a modified form of Peano's axioms, through the rationals to the theory of Dedekind sections and the irrationals, and finally to the arithmetic theory of the complex numbers. He set himself the somewhat tedious task of filling this gap, and, being Landau, he did the job incisively and once for all. Fashions change in mathematics, as in other domains, but it will be a very long time before this particular piece of work will need re-doing.

The production is in the usual clear and pleasing Chelsea style. T.A.A.B

Interpolation. By J. F. STEFFENSEN. 2nd edition. Pp. ix, 248. $3.50. 1950. (Chelsea Publishing Co., New York)

On the first appearance of the English version of this book, D. C. Fraser devoted a long review (*Gazette*, XIV, pp. 235–243) to a full discussion of its merits. He recalled the long Danish tradition of interpolation studies, with Oppermann, Thiele and Steffensen spanning three generations, culminating in Steffensen's appointment to a specially created chair of actuarial science in the University of Copenhagen ; and he hailed the present book as a " landmark in the history of the subject ".

Starting from scratch, the author deals with formulae of interpolation, construction of tables, inverse interpolation, summation formulae, the symbolic calculus, interpolation with several variables, in a clear, elegant and rigorous manner, though with that austere conciseness necessary in order to cover so much ground in 250 pages. This means that reading can not be casual, and the student must be prepared occasionally to think and work

for himself. He will be rewarded by a comprehensive view of the whole field. Of course, the subject has progressed since 1925, and on those aspects of interpolation which particularly bear on table-making, much could be added: we need only mention, for instance, Comrie's contributions. Nevertheless, Steffensen's book is still a classic account which no serious student can afford to neglect, and the Chelsea Company are to be thanked for making available once more as one of their series of reprints. T. A. A. B.

Irrationalzahlen. By O. PERRON. 2nd edition (rep.). Pp. viii, 199. $3.25. 1948. (Chelsea Co., New York)

A review of this admirable book appeared in Vol. XXIV, No. 259, p. 137, fairly soon after the publication of the second edition in Berlin. This present notice, therefore, need do little but call attention to the fact that it is now available as a reprint, dated 1948, and produced by the Chelsea Publishing Company of New York. This enterprising organisation is to be congratulated on having returned to circulation this work and many others of its kind, which might otherwise have remained virtually inaccessible for an indefinite period of time. They are also to be commended on the excellent reproduction and clear printing. J. H. PEARCE.

Vorlesungen über die Theorie der Algebraischen Zahlen. By ERICH HECKE. Pp. viii, 266. $3.95. 1948. (Chelsea Publishing Co., New York)

Einführung in die Elementare und Analytische Theorie der Algebraischen Zahlen und der Ideale. (2te. Auflage.) By EDMUND LANDAU. Pp. viii, 147. $2.95. 1949. (Chelsea Publishing Co., New York)

These are welcome reprints of two classics of algebraic number-theory: Hecke's book appeared in 1923; the first edition of Landau's in 1917 and the second in 1927. The two books have worn well and are complementary. Hecke leads gently up to the quadratic reciprocity law for general fields and a glimpse of class-field theory, whereas Landau is interested in the development and application of the analytic theory of the Dedekind zeta-function.

Diophantische Approximationen. By J. F. KOKSMA. Pp. vi, 157. $3.50. 1948. (Chelsea Publishing Co., New York)

The Chelsea Publishing Company have done a great service to all number-theoreticians by reprinting this indispensable handbook, which first appeared in the "Ergebnisse" series in 1936. It is a model of succinct presentation, and combines a summary of the principal developments with an exhaustive account of the literature. We hope that Professor Koksma will find time to compile an account of subsequent work with equal virtuosity. J. W. S, C.

The Theory of Groups. By H. ZASSENHAUS. Translated by S. KRAVETZ. Pp. viii, 159. $3.50. 1949. (Chelsea Publishing Co., New York)

Zassenhaus's book on group theory, first published in 1937, bears witness, to some extent, to the impact of van der Waerden's work on the exposition of algebra. For the reasonably sophisticated student, able to profit from a concise, closely-woven style, Zassenhaus's volume ensures a penetration to the core of the theory of groups, a somewhat notable achievement for a slim volume of 160 pages. The first chapter is introductory in character though not altogether in style, but in the second the heart of the book is reached, a chapter on homomorphic correspondence and groups with operators, round which the whole theory is arranged, with a consequent gain in logical order and connection. The consistent use of the idea of a homomorphic mapping, due largely to Emma Noether, binds the subject together very firmly. In

the remaining three chapters, a great many theorems of basic importance are derived by crisp, powerful arguments.

The treatment is hardly one for the novice. But a well-prepared student, with a taste for abstract algebra, would probably find this account a quick if not entirely easy road to the key positions of elementary group theory. The translation is rather stiff, but clear: the Chelsea Company have added another valuable item to their series of reprints and translations. T. A. A. B.

Theorie der Endlichen und Unendlichen Graphen. By Dénes König. Pp. 258. 1950. (Chelsea Publishing Co., New York)

This book was published in Leipzig in 1936, and the present reprint represents another addition to the long list of German texts issued by the Chelsea Publishing Company. It is the only systematic account of the Theory of Graphs, and the author has himself made a major contribution to the theory. The subject-matter forms a branch of combinatorial topology, and has applications in many branches of mathematics, including logic, the theory of games group theory and the theory (combinatorial) of matrices and determinants. The bibliography (up to 1935) is extensive. L. S. G.

Exercices de Mécanique. Par H. Beghin et G. Julia. Tome I, Fascicule I. Deuxième édition. Pp. vii, 338. 800 F. 1946. (Gauthier-Villars) Tome II, Fascicule II. Deuxième édition. Pp. 339–583. 1951. (Gauthier-Villars)

Once again, the French have made a very worthy addition to their already long list of excellent textbooks in mathematics.

At present under review are the two parts (published separately) of the first volume of a work (which contains two volumes) consisting of worked examples in mechanics. The examples are arranged under chapter headings, and each group contains a résumé of the theory essential to the solution of the problems in that group.

There are six chapters in the first part of the first volume, ranging from vectors and kinematics to moments of inertia and kinetics. In all, 223 examples are discussed in these chapters.

In the second part of the first volume there are again six chapters, starting with the fundamental laws of motion, and eventually dealing with work, power, the theorem of virtual work, impulses, and finally, there is a chapter devoted to Lagrange's equations, including their adaptation to problems concerning impulsive motions. There are altogether 64 exercises worked out in this section, the questions being longer and more difficult than those in the first part.

With regard to the questions, some are classical, and therefore well known, whilst others are taken from diverse sources, including the treatises of Routh and Appell, and the French Master and Fellowship examinations. In each chapter the examples have been arranged in the order of increasing difficulty, the most difficult being denoted by an asterisk. They are brought up to date by the fact that vector theory is used wherever possible.

The style of the work is excellent and the diagrams beautifully clear. In fact, I can thoroughly recommend this first volume to all students reading for an honours degree in mathematics. J. Williams.

Analytical and Applied Mechanics. By G. R. Clements and L. T. Wilson. Third edition. Pp. xi, 463. 47s. 1951. (McGraw-Hill)

This is essentially an elementary textbook on mechanics written by two professors (one retired), in the department of mathematics of the United States Naval Academy. The text develops from first principles the essence

of statics and dynamics separately, the former being treated first, followed by the section on dynamics and then, in conclusion, we have four chapters on stress and strain problems.

There is nothing original in the presentation of the subject-matter. In fact, it is just another textbook with the difference that it gives me the impression of trying to do as much as possible in the shortest possible space. For this reason and for reason of the style of the questions, I cannot recommend it as a textbook to students studying for English mathematical examinations ; it would be far better suited to the requirements of the student of engineering.

J. WILLIAMS.

The Hodograph Method in Gas Dynamics. By A. G. GHAFFARI. Pp. iv, 129. 1950. (Taban Press, Tehran)

This book contains a good elementary mathematical account of the Hodograph Method. The technique has been developed and used in the theory of dynamics of compressible fluids, and has proved to be very fruitful. A preface by Professor G. Temple, F.R.S., briefly reviews the contents, scope and value of the book.

The account is more or less self-contained and makes appropriate reference to Classical Hydrodynamics, Thermodynamics, the pioneer work of Chaplygin and that of other investigators. The main purpose of the book is to review the principal advances in this field which were made in Great Britain, U.S.A. and Germany during the period 1939–1945, when the method was carefully investigated because of its possible importance in connection with the theory of gas flow at speeds approaching that of sound.

The presentation, which embodies sections of Dr. Gaffhari's thesis for the degree of Ph.D. of the University of London, emphasises the use of methods to shorten numerical calculations and to achieve results within the accuracy required by engineering practice.

The book itself is clearly printed, but contains a number of minor defects in style and typography ; it is, however, a praiseworthy effort for a press which, presumably, does not have at its disposal as many different founts as we are accustomed to use for mathematical books printed in this country.

L. R.

Basic Mathematics of Technology. Vol. I. By J. CHANCE and G. F. SIMS. Pp. viii, 264. 8s. 6d. 1950. (University of London Press)

" This is the first of two volumes planned under this title." " It aims, while being directly pointed to the special needs of students taking technical courses, to give at the same time a broad and thorough training in the basic mathematical ideas and skills upon which the success of later work so much depends."

The book opens with an introduction to coordinates, and on the second page we are discussing ordinates and coordinates, vertices and major axes, and almost immediately we are finding the areas of what, to the beginner, must at first appear to be complicated figures. Algebraical symbols are introduced in Chapter 2. Within a few pages and with the help of many sketches we are well into the subject of algebra and building up formulae as a preparation for the consideration of the areas of " plane figures ". The common balance is convincingly used to illustrate the idea of the equation, and the insistence upon giving the statement of the process for each step in the early stages of solving equations should assist clear thinking. The chapters on graphical representation and on the positive and negative numbers are very clearly illustrated.

Under the heading " Geometry " are discussed angles, simple constructions,

triangles, parallels, polygons, the circle and the areas of circles and of various figures, of which the circle is a part, taken mostly from engineering practice. " Pythagoras' Rule " is well illustrated, and its application is demonstrated in a large number of well-selected exercises. " Direct proportion " graphs lead to gradients and the first trigonometrical consideration, the tangent of angles.

There is a good chapter dealing with indices and logarithms, and at this stage, including the determination of roots and powers, the written expression is as a power of 10. This volume ends with a chapter on the Mensuration of solids, most of which may be found in the workshops, two chapters continuing trigonometry to the solution of right angle triangles, and finally one on the graphs of formulae. To serve for revision there are 30 pages of further examples, all well selected.

" A special feature of the book is the liberal provision of diagrammatical material, both as illustrations and as a source of data for many of the exercises."

While the opening may perhaps embarrass the beginner, on working through the book one feels a freshness of an attack with a purpose, and as the authors say, " the emphasis throughout (the book) is on formula, the graph and and computation skills as the necessary tools of the subject ". The pupil or student, who has worked conscientiously through the book as the authors intend, should be a good example of their object and intention. The text, printing and the figures—the whole is well and clearly produced. E. J. A.

Model Papers in Algebra. By M. Y. DESHPANDE. Pp. v, 216. Rs. 3. 1949. (N.M.V. High School, Poona)

Forty-five " model papers " and six Bombay University Matriculation papers are preceded by five pages of the " essentials of Algebra ", viz. formulae and elementary rules. There are 102 pages of comprehensive hints to the solutions and answers, and a classified index of types. Subjects of questions include square and fourth roots, ratio and proportion, variation and graphs (linear or of the form $y=mx^2$). English teachers might think some of the factorisations, fractional equations and identities rather hard for O pupils. The " reconstruction " questions in multiplication and division are interesting.

B. A. S.

Trigonometry. By A. PAGE. Pp. viii, 276. 18s. 1951. (London University Press)

When a book " is designed to cover the requirements of students taking examinations up to University Scholarship level ", by what standard should it be judged? for there is the Cambridge Mathematical Scholarship, and the scholarship in which mathematics is taken as one of a group of subjects in which perhaps it plays a subsidiary part. Let it be said at once that for the former this book will require supplementation, as Dr. Page implicitly recognises by footnote references to Hardy and Knopp ; moreover, he points out (in other words) that the mere convergence of the Maclaurin series for $f(x)$ does not ensure that its sum is $f(x)$, and that, in finding the infinite product for $\sin x$, " the process of letting n tend to infinity requires justification ", but neither point is pursued. The convergence of infinite series involving complex arguments is nowhere mentioned, and such series are used for summations like $\sum_{1}^{\infty}(-)^{n-1}(\cos 2n\alpha)/n$ without reference to the exceptional case when $\alpha=\pm(2p+1)\pi/2$. It is difficult to know what knowledge the reader is assumed to possess : questions set require a knowledge of vectors, Argand diagram, Maclaurin's theorem and the hyperbolic functions, none of

which topics is mentioned in the text, though de Moivre's theorem and the exponential expressions for sin x and cos x are proved. On p. 46 a question assumes the " horizontal equivalent " method of map measurement, which is explicitly stated only on p. 84 ; on pp. 181, 182 much work is done involving declination before the term is defined on p. 183, and " subsolar " point (p. 180) is never defined.

The book has a strongly aeronautical tincture, much attention being paid to such problems as the effect of wind on aircraft navigation, the wind-lane method, the three main types of position lines, determination of wind speed and direction by the double drift method, determination of winds in the upper atmosphere, and astro-navigation. Good features are the explanations—necessarily brief—of the principles of map-making, of the usual map projections (formulae for the Mollweide and Mercator projections are given without proof) ; there are exercises on measuring the distances of the sun, moon and stars, on totality of eclipses, the length of the day, sunset and twilight, and the graduation of sundials. The only spherical formula used is the cosine rule, of which a proof is given. There is also work on rhumb-line and great circle sailing and numerous three-dimensional problems. A chapter is devoted to an elementary method of constructing trigonometrical tables.

The book opens with the ratio definition of tan θ and goes through the usual analytical trigonometry up to the power series for sin x, cos x, tan x, $\sin^{-1} x$, $\tan^{-1} x$, followed by the expansions of sin $n\theta$, cos $n\theta$, tan $n\theta$, factors of polynomials, finite and infinite products. The sine and cosine rules are first developed for acute angles, later for the general angle ; the " projection " proof of the compound angle formulae depends on rotation of axes : there is no separate chapter on properties of triangles and quadrilaterals, the formulae being scattered through the book : the solution of the cubic is not given. The student will need considerable help with some of the exercises, and would probably appreciate an explanation of the astrolabe depicted on p. 4.

I feel that there is an air of rush about most of the bookwork and, personally, I dislike the sandwiching of bits of bookwork in the middle of a set of exercises. Question 32 (i) on p. 20 is wrong. B. A. S.

Professional Opportunities in Mathematics. A Report for undergraduate students of mathematics, prepared by a committee of the Mathematical Association of America, printed in the *American Mathematical Monthly* (LVIII, 1, January 1951), and obtainable separately from Professor H. M. Gehman, University of Buffalo, Buffalo 14, N.Y., at 25 c. for single copies or 10 c. each for orders of ten or more.

This Report covers a field outside the range of our own Association's official activities. It deals mainly with opportunities in statistics, in industrial and government laboratories, and in the actuarial profession, and concludes with a list of 30 (American) books for further reading. But its two introductory pages, and the three on the teacher of mathematics in schools and universities, contain ideas that we should heed. We cannot maintain our subject in Modern and Technical schools without relating it to the lives of our pupils as they are lived and as they may have to be lived.

Our Reports consider the teaching of mathematics almost entirely with the view of cultivating more mathematicians ; we have not yet begun a report on mathematics as it might be used in all branches of applied or social sciences. This Report urges " the prospective teacher to take as much work as he can in other sciences, as well as in engineering, economics and the social sciences ". Those who would go in for an industrial or a university career are warned, however, that university mathematics is something more than fun in solving problems, and that the Ph.D. is a necessity for all interesting industrial jobs

as well as for success in a university career. But " the mathematician, to be effective in industry, government or university, must be a well-rounded person "—a phrase to which our leaner members may object. J. T. C.

A Manual for the Slide Rule. By PAUL E. MACHOVINA. Pp. 78. 6s. 6d. 1950. (McGraw-Hill)

He who writes a textbook on some branch of elementary mathematics—with the exception of Dr. Lietzmann—must expect to be shot at. The task should be to leave no stone unturned on familiar plots, no path unexplored in a well-mapped landscape, yet the reason for publication is surely some originality in treatment, some fresh angle of view. Mr. Machovina's booklet satisfies on the first two points, and puts up a good case on the third.

The introductory sections are well chosen and brief. The thumbnail history is a good feature : to how many of the innumerable users of logarithms and slide rules are the names of Napier and Briggs, Gunter and Oughtred unknown, Mannheim a synonym for the instrument itself? Mr. Machovina might have tackled more boldly his exposition of the nature of logarithms by abandoning the conventional 10^x introduction, and underlined his history section by going back to Napier. The treatment he gives is usual with writers of elementary texts, for whom Nunn and Fletcher have lived in vain. How long must our alumni endure such " explanations " as : " Numbers commonly used as bases for systems of logarithms are 10 and the irrational (*sic!*) number e (2·718 . . .) "?

The sections dealing with the slide rule operations are well written and clearly illustrated. A short chapter on accuracy is a welcome thought : Mr. Inman would approve, but some of his customary lucidity should have been borrowed by the author to explain that the product of 11·37 and 2·42 lies between 27·47 ... and 27·58 ... , instead of quoting " Holman's rule ". This, while important, is not sufficiently the product of original thought to entitle Mr. Holman to immortality. Section XVI, " On Locating the Decimal Point," should be made a topic of compulsory study to intending writers of elementary works on arithmetic, and recommended to intending teachers of the same. In sub-section 88 I find the author at fault : my own Faber, a pre-war Mannheim, has an *ST* scale. Possibly the remark applies to American makes.

In view of the trend of price levels this booklet, printed on good paper in clear type, but bound in limp covers, is not exorbitantly priced at 6s. 6d. A penny a page is good value to the beginner, but one would have welcomed a short section on other types of slide rule in place of the perforated problem sheets. At this price one copy ought to serve several relays of students, and while Mr. Machovina does well to disagree with Mr. Ford on the importance of history, his publishers would do well to follow in those illustrious footsteps in the matter of sales policy. As it is, Student No. 2 will not only lose his problem sheets ; he will not know what he has missed. J. E. B.

The Knowall Maths. I, II. By D. PONTON. Pp. 94, 2, 20 ; 183, 54, 97. 7s. 6d. ; 10s. 1950. (11 Churchfield Road, Poole, Dorset)

This is a series of four books providing a Complete Course in Arithmetic, Algebra, Geometry and Numerical Trigonometry from the Nursery to the University Standard.

In the early sections of the first volume, the author has introduced the young child to the world of number by means of stories which can be dramatised. The material provided is for the teacher to reproduce for the pupil, but it has been clearly set out, so that the child himself can understand it, provided he can read the text. Later on the method for working formal examples is explained in detail step by step, and several solutions illustrate

each process. There is, however, need for many more exercises for the pupils' own use. The author suggests that any suitable book of examples should be used to supplement these given. No attempt has been made to do more than briefly introduce Algebra, but I feel that even such slight treatment would be more appropriate to Book II.

The section on Geometry is confined to the various definitions concerning angles, triangles and quadrilaterals, and is introduced by means of stories and humorous sketches, and there is an attractive summary containing the essentials for the pupil's notebook.

In Book II the section on Arithmetic has been divided into three distinct parts: Whole Numbers and Decimals, Money, Weights and Measures—including the Metric System—and finally Fractions, Ratio and Proportion. More emphasis has been given to Decimals, in that they are taught first, and the pupil's first introduction to fractions is in order to convert them to decimals, before any skill in handling them has been acquired. More arithmetical problems are given in this volume, and there are convenient tests at the end of each section.

In dealing with algebra, the author has followed up the elementary rules with chapters on the simple equation, transformation of formulae, and the work on graphs is developed so far as velocity-time graphs. In this subject, in particular, there are insufficient examples on each new rule.

The use of mathematical instruments, and some experimental work, is covered in the first part of the section on Geometry. The other two parts, however, consist of formal work covering 20 theorems and 15 constructions. I think more experimental work might have been introduced here, in view of the age of the pupils.

These two books are not in general suitable for the State Primary Schools, since the standard attained is far higher than is expected of the average eleven-year-old. However, they could be used with advantage by a governess, or a teacher dealing with small groups of highly selective children. Y. B. G.

CORRESPONDENCE.

POLYHEDRON NOMENCLATURE.

To the Editor of the *Mathematical Gazette*.

Dear Sir,—In a footnote to page 144 of his edition of Ball's *Mathematical Recreations and Essays* Prof. H. S. M. Coxeter drew attention to a mistake in the present edition of the *Encyclopaedia Britannica* where the attributes of the great dodecahedron are assigned to the small stellated dodecahedron and *vice versa*. Unfortunately this error has crept into the pages of Lines' *Solid Geometry* and from thence into Mr. C. Hope's recent paper on " The Nets of the Regular Star-faced and Star-pointed Polyhedra " (*Math. Gazette*, Vol. XXXV, p. 8). The legend to Lines' Fig. 117, p. 171, should read " Small stellated dodecahedron " and that to Fig. 119, p. 173, " Great dodecahedron ". Photographs of models of these solids are given in Plate II, Figs. 31 and 34 respectively, of Coxeter's edition of Ball mentioned above: a model of the great dodecahedron is also depicted in *Chambers's Encyclopaedia*, 1950, Vol IX, p. 152, art. " Mathematical Models ".

It may be worth while to add that the names by which Poinsot's polyehdra are known were given to them by Cayley in 1859 (*Phil. Mag.*, [4], Vol. XVII, p. 123).

Sidney Melmore.

PRINTING CONTRACTIONS.

To the Editor of the *Mathematical Gazette*.

SIR,—In connection with the style of printing contractions used in Quenouille's book, of which you print a review on page 287, I think the attention of teachers of mathematics should be drawn to a recommendation of the Scottish Council for Research in Education published on pp. 10, 11 of their 22nd *Annual Report* (1949–50). It runs as follows :

" *Periods*

" In the punctuation of contractions or abbreviations the policy, advocated by H. G. Fowler in *Modern English Usage*, for words the contracted form of which ends with the same letter as the uncontracted form, of omitting the period should be adopted : " Mr " instead of " Mr.", " hr " instead of " hr.", " yd " instead of " yd.", " qr " instead of " qr.". This principle should be extended to such terms as " min ", " lb ", " oz ".

" Even where a contracted term forms an unabbreviated word, " number —no ", " inch—in ", the period should be omitted as the context usually indicates the meaning.

" *Plurals*

" The plurals of contracted arithmetical terms, as advised by *Rules for Compositors and Readers*, should be written without the " s " ; for instance, lb, oz, cwt, sec ; and metric system terms—cm, gm—should be used for both singular and plural."

I venture to submit that in some cases, such as the one I refer to in my review, adoption of this recommendation may not add to clarity in mathematical texts. I have seen no reference elsewhere to this recommendation, and would be glad to know what is the feeling among other teachers of mathematics, particularly those teaching at the most elementary level.

Yours, etc., FRANK SANDON.

UNITS IN DYNAMICS.

To the Editor of the *Mathematical Gazette*.

SIR,—In Mr. Welch's letter, *Mathematical Gazette*, No. 309, p. 181, he says that one school of thought is in favour of using gravitational units at first *without too much insistence on accurate terminology*.

He has no justification whatever for the words I have put in italics. I, like the others who advocate the early use of gravitational units, feel most strongly the importance of insisting on accurate terminology.

With beginners it seems wise to use the same units in Statics and Dynamics. Much trouble is bound to arise if in Statics we talk of a force of M gm. ; M gm. is essentially a mass, the force is the weight of that mass, *i.e.* M gm.wt. I strongly urge that in both Statics and Dynamics care should be taken to speak of a force of M gm.wt., and to measure work in ft.lb.wt. (not in ft.lb.).

What muddled thought, or muddled teaching, can have led to such solutions as Mr. Welch gives? The pupil who has been brought up on gravitational units would write :

Let P gm.wt. be the required force.

Then $P - \mu M$ gm.wt. produces α cm./sec.

But M gm.wt. produces g cm./sec.

$$\therefore \frac{P-\mu M}{M} = \frac{\alpha}{g}.$$

$$\therefore P = \mu M + M\frac{\alpha}{g}.$$

ST. ANDREWS MATHEMATICAL COLLOQUIUM

18th–28th July, 1951

A feature of pre-war British Mathematics was revived in St. Andrews this summer when the Edinburgh Mathematical Society organised a successful Mathematical Colloquium. The meetings were held and the majority of the hundred members were accommodated in University Hall.

Courses of lectures were given by Prof. H. S. M. Coxeter, F.R.S., of Toronto, on " Kaleidoscopes and Quadratic Forms " ; by Prof. A. Erdélyi of Pasadena on " The Analytic Theory of Systems of Partial Differential Equations " ; by Mr. A. E. Ingham, F.R.S., of Cambridge, on " The Analytical Theory of Numbers " ; by Prof. J. L. Synge, F.R.S., of Dublin, on " The Geometry of Function Space " ; and by Prof. G. Temple, F.R.S., of London, on " Mathematical Problems of Supersonic Flow ".

The Colloquium was opened by its President, Prof. H. W. Turnbull, F.R.S., formerly Professor in St. Andrews, who gave a stimulating lecture on an historical subject. Single lectures were also given by Prof. L. E. J. Brouwer of Amsterdam on " The Influence of Mathematics on Logic " and by Prof. Sophie Piccard of Neuchatel on " The Theory of Groups ". A feature of the Colloquium was the unusually high standard of exposition provided by the lecturers.

Several mathematicians were accompanied by their relations and friends. These joined in the numerous social occasions, which included visits to the University Library and Observatory, musical and theatre evenings, 'bus excursions and Scottish country dances. The hope was expressed by many of those present that the St. Andrews Colloquium would again become a regular event, as in the years before the war.

LEICESTERSHIRE BRANCH

Report for 1950

In the autumn term the number of Branch members who are members of the parent Association rose to thirty, and the Branch was invited to nominate a representative on the Council of the Association.

Four meetings were held during the year at University College, Leicester. On the 9th March, 1950, Mr. R. W. Bonham of the Worcester School for the Blind, himself a blind man, spoke of the important part which mathematics can play in the life of a blind person, and described the special problems of the blind pupil and teacher. After the lecture members examined the teaching equipment and table books in use in schools for the blind, and learned to appreciate the true significance for mathematics of that much used term *visual aid.*

On the 16th June Mr. A. W. Riley spoke of the tasks which face the teacher of mathematics in a modern school. He expressed the hope that the modern school would use its freedom from examination cares to stress the beauty as well as the utility of mathematics.

The Annual General Meeting was held on 24th November, and afforded a number of speakers an opportunity to compare the Syllabuses of the several Examining Bodies for the General Certificate of Education.

The last meeting of the year was held on 11th December when Mr. E. J. F. Primrose spoke on the mathematical determination of Easter Day, and discussed the source of the small discrepancies between the dates given in the Prayer Book and the general formula due to Gauss.

Officers. President and Chairman, Professor R. L. Goodstein ; *Treasurer,* Mr. L. G. Clarke ; *Secretary,* Mr. R. H. Collins ; *Council Representative,* Mr. R. H. Collins.

SOUTHAMPTON AND DISTRICT BRANCH

THE Southampton Branch had a successful year, the number of full members being over 30, together with a considerable body of associate and student members. Mr. R. H. V. Roseveare was succeeded as President by Mr. W. Hartas Jackson, and Dr. F. G. Maunsell continued as Secretary. Seven meetings of the Branch were held during the year.

On October 20, 1950, Mr. G. L. Parsons (the former Secretary of the Association) gave a talk on " Mathematics and History ". This was very enjoyable, and we have persuaded Mr. Parsons to arrange to visit us again next year.

On December 6, 1950, Dr. Rankin read a paper on " Change Ringing ". This was interesting, both for the insight into a fascinating pursuit and for the applications of group theory to the problems.

On February 1, 1951, Dr. Topping (whom we have all known as Programme Secretary) talked on " Approximations ". His talk was concerned particularly with methods of introducing the ideas of approximations to technical students.

On February 16, 1951, a discussion on the *Trigonometry Report* took place. We were fortunate enough to persuade Mr. Tuckey and Mr. Manisty (the Chairman and Secretary of the *Report* Sub-Committee) to attend, and the discussion was most illuminating.

On March 2, 1951, Dr. Mullineux (of University College, Southampton) read a paper on " Geometry and Number ", in which he outlined some of the results of the theory of quadratic forms.

On May 25, 1951, the Branch met at Winchester College. We have held one meeting a year at Winchester instead of Southampton for the last three years, and members all greatly appreciate the privilege of holding a meeting at this historic school. On this occasion we had an interesting talk by Dr. C. A. B. Smith of the Galton Laboratory on " Mathematical Problems in Genetics ".

On June 15, 1951, Professor Richards, recently appointed Professor of Aeronautical Engineering at Southampton, threw open the Aeronautical Laboratory there to members. It was most interesting to see the wind tunnels and other apparatus, and to hear about the various lines of research which are being carried on.

It is believed that the majority of full members of the Association who live near Southampton are members of the Branch. The Secretary would be very much obliged if any who are not, or any who come to reside near Southampton, would get in touch with him at University College, Southampton.

F. G. MAUNSELL, *Hon. Secretary.*

THE LIVERPOOL MATHEMATICAL SOCIETY
LIVERPOOL BRANCH OF THE MATHEMATICAL ASSOCIATION

REPORT FOR THE SESSION 1950–1951

Officers. President, Miss J. S. Batty ; *Vice-President,* Mr. A. T. F. Nice ; *Secretary,* Mr. J. Kershaw ; *Treasurer,* Mr. L. Sowerby ; *Auditor,* Miss W. Taylor ; *Committee,* Dr. W. B. Bonnor, Mr. W. J. Fairhurst, Miss M. H. Greig, Dr. C. W. Jones, Miss W. Taylor, Miss I. Thackara, Mr. E. D. Camier.

Activities.

The Society held seven meetings of which reports are given below. Although meetings were well attended some recession occurred in the paid membership totals. The Society's Mathematical Prize for 1950–1951 was awarded to S. Lipton.

Reports of Meetings.

23rd October, 1950. The Presidential Address was delivered by Miss J. S. Batty, M.Sc., of the University, on the subject " Perfect Numbers ". After defining the above class of numbers, their relationship to Mersenne Primes was made clear and the speaker deduced certain of the properties of even perfect numbers of Euclid's form. Lucas' Test for the primality of Mersenne numbers and H. S. Uhler's work on the composite nature of some M numbers were both covered in the course of this talk.

13th November, 1950. An address entitled " Problems in Cosmology " was given by W. B. Bonnor, Esq., B.Sc., Ph.D. of the University. From the principal data available to cosmologists the conclusion was drawn that the universe is expanding. Using the methods of General Relativity various models of the Universe were derived but shown in the main to suffer from defects of either inadequacy of time-scale or of requiring the introduction of special hypotheses.

4th December, 1950. Professor M. J. Lighthill of Manchester University, gave a talk on " Aerodynamic Noise ". Making a mathematical analysis of the problem the speaker suggested that sound emanating from fully developed turbulent flow constitutes a quadruple field, that its intensity varies as (nearly) the eighth power of a typical velocity and that the greatest noise derives from where heavy mean shear stress coincides with a high level of turbulence. A silencer for jet engines based on the above considerations was being given experimental tests.

22nd January, 1951. Under the Chairmanship of Professor L. Rosenhead, F.R.S. of the University, Liverpool, a discussion was held on the subject " The Scope and Significance of Mathematics To-day ". Four members of the University staff introduced the subject and, following contributions from the body of the meeting, were allowed to reply to their critics. The Chairman, in summing up, stressed the essential unity embracing all mathematics, the increasing importance of the subject, and the value of mathematicians getting together to discuss their work.

5th March, 1951. J. T. Combridge, Esq., M.A., of King's College, University of London, spoke on the subject " The Place of the Teaching Committee in the Work of the Mathematical Association ". Mr. Combridge traced the historical development of the M.A. Teaching Committee and gave details of its present-day composition and organisation. An account was given of the procedure adopted in the preparation of the Reports of which an ambitious programme is now in hand.

21st April, 1951. A party from the Society visited the Liverpool Tidal Observatory at Bidston, Birkenhead, by the courtesy of the Director, Dr. A. T. Doodson. The astronomical, meteorological, seismic and tidal equipment were inspected. Two remarkable tidal prediction machines were demonstrated and also a model for solving the equations of tidal flow in the River Thames.

21st May, 1951. The Annual General Meeting commenced with the consideration of the Treasurer's Interim Report and the election of the new Council. The usual business of the meeting was followed by an Address on " Tradition and Text-books in the Teaching of Mathematics " given by R. H. Cripwell, Esq., B.A., of Didsbury Training College. Mr. Cripwell used the text-books of different periods to illustrate the development of mathematical teaching particularly during the present century. He considered that there was much room for improvement in Secondary School teaching, that much unessential algebraic manipulation could be excluded and that the functional value of mathematics should receive greater emphasis.

Branch Representation.

The branch is represented on the Council of the Mathematical Association. The Secretary attended the meetings of this body convened during the 1950-1951 Session.

J. Kershaw, *Hon. Secretary.*

BOOKS RECEIVED FOR REVIEW.

S. Banach. *Mechanics.* Translated by E. J. Scott. Pp. iv, 546. \$6. 1951. Monografje Matematyczne, 24. (Warsaw : Stechert-Hafner, New York)

A. R. Bielby. *A new geometry and trigonometry.* Pp. viii, 448. 10s. 6d ; without answers, 10s. Part I : 6s. 6d. Part II : 6s. 6d. 1951. (Longmans Green)

B. C. Brookes and **W. F. L. Dick.** *Introduction to statistical method.* Pp. viii, 288. 21s. 1951. (Heinemann)

J. W. Cell. *Analytic geometry.* 2nd edition. Pp. xii, 326. 30s. 1951. (John Wiley, New York ; Chapman and Hall)

J. H. Ebbutt. *Shapes and sizes. III.* Pp. 33. 1s. 3d. With answers, 1s. 6d. 1951. (Black)

G. Engel. *Analytische Geometrie.* Pp. viii, 239. DM. 18. 1950. (Walter de Gruyter, Berlin)

W. L. Ferrar. *Finite matrices.* Pp. vi, 182. 17s. 6d. 1951. (Oxford University Press)

P. Finsler. *Über Kurven und Flächen in allgemeinen Räumen.* Rep. Pp. ix, 160. Sw. fr. 12 ; geb. sw. fr. 14.80. 1951. (Birkhäuser, Basel)

S. L. Green. *Intermediate dynamics and statics.* Pp. 298. 12s. 6d. 1951. (University Tutorial Press)

P. R. Halmos. *Introduction to Hilbert space and the theory of spectral multiplicity.* Pp. 114. \$3.25. 1951. (Chelsea Co., New York)

G. Hoheisel. *Gewöhnliche Differentialgleichungen.* 4th edition. Pp. 129. DM. 2.40. 1951. Sammlung Göschen, 920. (Walter de Gruyter, Berlin)

A. Hooper. *The River Mathematics.* Pp. 370. 18s. 6d. 1951. (Oliver & Boyd)

Sir Charles Inglis. *Applied mechanics for engineers.* Pp. xii, 404. 42s. 1951. (Cambridge University Press)

J. J. de Kock and **A. J. van Zyl.** *Senior mathematics.* Pp. xv, 371. 12s. 6d. 1951. (Maskew Miller, Cape Town)

E. E. Kramer. *The main stream of mathematics.* Pp. xii, 321. 30s. 1951. (Oxford University Press)

A. Linder. *Statistische Methoden für Naturwissenschafter, Mediziner und Ingenieure.* 2nd edition. Pp. 238. Sw. fr. 26 ; geb. sw. fr. 30. 1951. (Birkhäuser, Basel)

R. H. Macmillan. *An introduction to the theory of control in mechanical engineering.* Pp. xiv, 195. 30s. 1951. (Cambridge University Press)

A. Maroger. *Les trois étapes du problème Pythagore-Fermat.* Pp. 98. 400 fr. 1951. (Vuibert, Paris)

J. H. Murdock. *The teaching of mathematics in post-primary schools.* Pp. 183. 12s. 6d. 1950. (New Zealand Council for Educational Research ; London, Geoffrey Cumberlege)

H. E. Parr. *School mathematics. III.* Pp. viii, 239, xxii. 6s. 6d. ; without answers, 6s. 1951. (Bell)

P. Pasquier. *Initiation à l'étude du ciel.* Pp. 48, with 7 plates. 330 fr. 1951. (Vuibert, Paris)

H. B. Phillips. *Differential equations.* 3rd edition. Pp. viii, 149. 24s. 1951. (John Wiley, New York ; Chapman and Hall)

A. Pope. *Aerodynamics of supersonic flight.* Pp. xi, 185. 25s. 1951. (Pitman)

G. R. Rich. *Hydraulic transients.* Pp. x, 260. 51s. 1951. (McGraw-Hill)

M. F. Rosskopf, H. D. Aten and **W. D. Reeve.** *Mathematics : a first course.* Pp. 472. \$2.60. 1951. (McGraw-Hill, New York)

The required force is $M\left(\mu+\frac{\alpha}{g}\right)$ gm. wt.

Yours, etc., A. W. SIDDONS.

To the Editor of the *Mathematical Gazette*.

DEAR SIR,—Certainly it is true that teachers of mechanics are divided in their choice of the most appropriate system of units to use in introducing the subject. But is it not also true that there is a danger of this being decided for us by the compilers of our examination syllabuses? Moreover, they not only influence teaching directly, but also decide the emphasis of our elementary textbooks.

What justification is there for continuing to examine pupils in mechanics at the Ordinary level of the G.C.E., either as a separate subject or as part of a paper in " Additional Mathematics "? Few teachers are allowed more than 60 periods in which to teach the subject before the examination is taken. In that time they may either try to introduce the ideas of mechanics (and I regard such a course as being of great educational value to a pupil who is not intending to specialise in science or mathematics); or they may prepare for the examination. It is surely impossible adequately to do both.

If a teacher makes the attempt, however, he is virtually compelled to adopt the gravitational system of units, whatever his own preference. No boy at that age can be expected to master two different sets of equations; and at present the examiners have decreed that, although they may ask him to define a poundal or to distinguish weight from mass, he shall give his answers in lb.wt and in ft.lb. So long as the examination continues, we shall be expected to enter our pupils for it. Is there not then a case for having two alternative syllabuses, one on the lines of the existing syllabus, the other based on the use of absolute units and involving a more fundamental treatment of dynamics?

Yours faithfully, D. A. QUADLING.

LAPLACE TRANSFORMS.

To the Editor of the *Mathematical Gazette*.

SIR,—In his review of *Transformation Calculus and Electrical Transients* by S. Goldman (*Gazette*, XXXIV, No. 309) Mr. H. V. Lowry deplores the fact that the author defines the Laplace transform of a function $f(t)$ by

$$\int_0^\infty e^{-st} f(t)\,dt$$

rather than by

$$p\int_0^\infty e^{-pt} f(t)\,dt.$$

In favour of the " p-method ", Mr. Lowry instances the fact that it transforms a constant into itself. However, in an elementary course which excludes the inversion integral, the extra p in the " p-method " adds considerably to the labour of splitting up the rational algebraic fractions arising into their partial fractions. On this account, the saving in time seems to leave the advantage with the " s-method ".

Yours, etc., M. HUTTON.

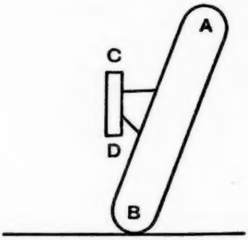
CAR WHEELS.

To the Editor of the *Mathematical Gazette*.

SIR,—The question posed by Professor Brown in the discussion on " The

Teaching of Mechanics ", quoted on p. 178 of *Gazette* No. 309, was a perfectly fair one : I do not believe, however, that the explanation given is correct.

Whether the turning of the front wheels raises or lowers the c.g. of the car depends not on the inclination of the wheels to the ground, but on the inclination of the pivot-axles, about which the wheels are turned when steering. The most important principle to bear in mind when designing the front wheels of a car is that the pivot-axle should meet the ground in the same point as the tyre (looked at from the front). So, as in Fig. 1, the wheel AB is made to lean out and the pivot-axle CD is vertical. The reason for this is so that there should not be an excessive torque tending to wrench the wheel off. The reaction of the ground at B is vertical, and so we require the axle to be vertically above B.



In order to obtain a certain amount of " castoring " effect (automatic straightening-out after a turn), another method is used. The pivot-axle is sloped forward (from 6° to 10°) so that it meets the ground in front of the point of contact of the wheel. It can be seen that this arrangement actually makes the c.g. *drop* when the wheels are turned, but this result is more than counterbalanced by the dynamical effect of the trailing of the point of contact behind the line of the pivot-axle, the frictional resistances at the point of contact giving couples which tend to restore the wheels to the straight.

One other point about the front wheels. The pivot-axle does not in general point exactly to the centre-line of the wheels, but slightly inside it. Consequently when going forward the wheels tend to splay out, and to counteract this they are made to " lead-in ", so that the front edges of the wheels are about $\frac{1}{2}$ inch closer together than the rear edges.

Yours, etc., F. G. Maunsell.

A PRIORITY REFERENCE.

Mr. C. E. Walsh writes to point out that the result in Note 2223 (*Gazette*, xxxv, p. 189) was proved by him in *Edinburgh Mathematical Notes*, No. 37, (1949), pp. 22-3.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW

ctly
t.
car
ina-
ing.
eels
the
ean
here
The
be

atic
e is
the
ally
han
act
on-

eral
nse-
ract
are

LL.

ette,
37,